1 Objective stress rates

Determine the stress-strain relation in a shear test

$$\boldsymbol{F} = \boldsymbol{I} + \dot{\gamma}_0 t \boldsymbol{e}_1 \otimes \boldsymbol{e}_2, \tag{1}$$

where the stress-rate–strain-rate relation

$$\dot{\boldsymbol{\sigma}} = \lambda \mathrm{tr}(\boldsymbol{D}) + 2\mu \boldsymbol{D} \tag{2}$$

should be used in conjunction with the Truesdell–, Zaremba-Jaumann– and the polar co-rotational stress rates,

$$\dot{\boldsymbol{\sigma}}^{\mathrm{TD}} = \dot{\boldsymbol{\sigma}} - \boldsymbol{L}\boldsymbol{\sigma} - \boldsymbol{\sigma}\boldsymbol{L}^T + \mathrm{tr}(\boldsymbol{D})\boldsymbol{\sigma} \tag{3}$$

$$\dot{\boldsymbol{\sigma}}^{\mathrm{ZJ}} = \dot{\boldsymbol{\sigma}} - \boldsymbol{W}\boldsymbol{\sigma} + \boldsymbol{\sigma}\boldsymbol{W} \tag{4}$$

$$\dot{\boldsymbol{\sigma}}^{\mathrm{polar}} = \dot{\boldsymbol{\sigma}} - (\dot{\boldsymbol{R}}\boldsymbol{R}^T)\boldsymbol{\sigma} + \boldsymbol{\sigma}(\dot{\boldsymbol{R}}\boldsymbol{R}^T). \tag{5}$$

The stresses are zero at $t = 0$.
Motivation: Sometimes it is useful to have an elastic law in terms of rates. For example in case of a coupling with a viscous fluid material law it may be convenient to have the elastic part of the problem as well in terms of the spatial strain rate $\boldsymbol{D}$. However, the spin $\boldsymbol{W}$ has to be accounted for explicitly: the Cauchy stresses rotate with the body. This is done by inserting the material law (eq. 2) into a co-rotational rate (eqs. 3 to 5). These are obtained by considering the time derivatives of material stress tensors or simply proposed, see Bertram, Elasticity and Plasticity of Large Deformations, end of Section 3.4. The resulting material laws are called hypo-elastic, which is more general than hyper-elasticity. The arbitrariness in selecting a co-rotational rate indicates a missing constraint, namely an integrability condition. We will see that this can lead to curious results, (firstly?) obtained by Dienes, 1979 (Acta Mechanica 32, 217–232).
Solution: With $\boldsymbol{F}(t)$ at hand we can easily calculate $\boldsymbol{L}$ and its symmetric and anti-symmetric parts $\boldsymbol{D}$ and $\boldsymbol{W}$,

$$\boldsymbol{L} = \dot{\boldsymbol{F}}\boldsymbol{F}^{-1} = \dot{\gamma}_0 \boldsymbol{e}_1 \otimes \boldsymbol{e}_2 (\boldsymbol{I} - \dot{\gamma}_0 \boldsymbol{e}_1 \otimes \boldsymbol{e}_2) = \dot{\gamma}_0 \boldsymbol{e}_1 \otimes \boldsymbol{e}_2 \tag{6}$$

$$\boldsymbol{D} = \dot{\gamma}_0/2(\boldsymbol{e}_1 \otimes \boldsymbol{e}_2 + \boldsymbol{e}_2 \otimes \boldsymbol{e}_1) \tag{7}$$

$$\boldsymbol{W} = \dot{\gamma}_0/2(\boldsymbol{e}_1 \otimes \boldsymbol{e}_2 - \boldsymbol{e}_2 \otimes \boldsymbol{e}_1) \tag{8}$$

The trace of $\boldsymbol{D}$ turns out to be zero. We now have everything at hand to write down the Truesdell rate of the Cauchy stresses,

$$\dot{\boldsymbol{\sigma}}^{\mathrm{TD}} = \mu\dot{\gamma}_0(\boldsymbol{e}_1 \otimes \boldsymbol{e}_2 + \boldsymbol{e}_2 \otimes \boldsymbol{e}_1) - \dot{\gamma}_0 \boldsymbol{e}_1 \otimes \boldsymbol{e}_2 \boldsymbol{\sigma} - \boldsymbol{\sigma}\dot{\gamma}_0 \boldsymbol{e}_2 \otimes \boldsymbol{e}_1. \tag{9}$$

We can give, with the symmetry of $\boldsymbol{\sigma}$, the components of $\dot{\boldsymbol{\sigma}}^{\mathrm{TD}}$,

$$\dot{\boldsymbol{\sigma}}^{\mathrm{TD}} = \dot{\gamma}_0 \begin{bmatrix} -2\sigma_{12} & \mu - \sigma_{22} & -\sigma_{23} \\ & 0 & 0 \\ & & 0 \end{bmatrix} \boldsymbol{e}_i \otimes \boldsymbol{e}_j. \tag{10}$$

We can identify a rather simple system of ordinary differential equations.

$$\dot{\sigma}_{11} = -2\dot{\gamma}_0\sigma_{12} \tag{11}$$

$$\dot{\sigma}_{12} = \dot{\gamma}_0(\mu - \sigma_{22}) \tag{12}$$

$$\dot{\sigma}_{13} = -\dot{\gamma}_0\sigma_{23} \tag{13}$$

$$\dot{\sigma}_{22} = \dot{\sigma}_{23} = \dot{\sigma}_{33} = 0. \tag{14}$$

With the initial condition for $\boldsymbol{\sigma}$, all stress components except σ_{11} and σ_{12} remain zero, while the nonzero components are obtained by time integration,

$$\sigma_{12} = \mu\dot{\gamma}_0 t \tag{15}$$

$$\sigma_{11} = -\mu\dot{\gamma}_0^2 t^2. \tag{16}$$

One can argue that, with $\gamma = \dot{\gamma}_0 t$, the result is reasonable as long as $\gamma/\gamma^2 \gg 1$ holds, i.e., $\gamma \ll 1$.
Next we consider the Zaremba-Jaumann rate, which gives

$$\dot{\boldsymbol{\sigma}}^{\mathrm{ZJ}} = \mu\dot{\gamma}_0(\boldsymbol{e}_1 \otimes \boldsymbol{e}_2 + \boldsymbol{e}_2 \otimes \boldsymbol{e}_1) - \frac{\dot{\gamma}_0}{2}(\boldsymbol{e}_1 \otimes \boldsymbol{e}_2 - \boldsymbol{e}_2 \otimes \boldsymbol{e}_1)\boldsymbol{\sigma} + \frac{\dot{\gamma}_0}{2}\boldsymbol{\sigma}(\boldsymbol{e}_1 \otimes \boldsymbol{e}_2 - \boldsymbol{e}_2 \otimes \boldsymbol{e}_1). \tag{17}$$

The components of $\dot{\boldsymbol{\sigma}}^{\mathrm{ZJ}}$ evaluate to

$$\dot{\boldsymbol{\sigma}}^{\mathrm{ZJ}} = \dot{\gamma}_0 \begin{bmatrix} -\sigma_{12} & \mu + \frac{1}{2}(\sigma_{11} - \sigma_{22}) & -\frac{1}{2}\sigma_{23} \\ & \sigma_{12} & \frac{1}{2}\sigma_{13} \\ & & 0 \end{bmatrix} \boldsymbol{e}_i \otimes \boldsymbol{e}_j, \tag{18}$$

leading to the following system of ODEs,

$$\dot{\sigma}_{11} = -\dot{\gamma}_0\sigma_{12} \tag{19}$$
$$\dot{\sigma}_{12} = \dot{\gamma}_0\mu + \frac{\dot{\gamma}_0}{2}(\sigma_{11} - \sigma_{22}) \tag{20}$$
$$\dot{\sigma}_{22} = \dot{\gamma}_0\sigma_{12} \tag{21}$$
$$\dot{\sigma}_{13} = -\frac{\dot{\gamma}_0}{2}\sigma_{23} \tag{22}$$
$$\dot{\sigma}_{23} = \frac{\dot{\gamma}_0}{2}\sigma_{13} \tag{23}$$
$$\dot{\sigma}_{33} = 0. \tag{24}$$

We see that $\sigma_{33} = 0$. The rest is solved by taking additional time derivatives and consecutive inserting to resolve the coupling, e.g., the time derivative of eq. (22) is inserted into eq. (23) to yield

$$\ddot{\sigma}_{13} = -\frac{\dot{\gamma}_0^2}{4}\sigma_{13}. \tag{25}$$

One recognizes the ODE of a harmonic oscillator, solved by

$$\sigma_{13} = C_1 \sin(\omega_0 t) + C_2 \cos(\omega_0 t), \qquad \omega_0 = \frac{\dot{\gamma}_0}{2} \tag{26}$$

With the initial conditions $\sigma_{13} = \sigma_{23} = 0$ we find $C_1 = C_2 = 0$. The same result is obtained for σ_{23}. Similarly, we can take the time derivative of eq. (20) and replace $\dot{\sigma}_{11}$ and $\dot{\sigma}_{22}$ by eqs. (21) and (19),

$$\ddot{\sigma}_{12} = \dot{\gamma}_0^2\sigma_{12}, \tag{27}$$

thus

$$\sigma_{12} = C_3 \sin(\dot{\gamma}_0 t) + C_4 \cos(\dot{\gamma}_0 t). \tag{28}$$

With $\sigma_{12}(0) = 0$ the coefficient $C_4 = 0$. However, with $\dot{\sigma}_{12}(0) = \mu\dot{\gamma}_0$ the coefficient C_3 is evaluated to $C_3 = \mu$, resulting in

$$\sigma_{12} = \mu \sin(\dot{\gamma}_0 t). \tag{29}$$

Inserting this in eqs. (21) and (19) and integrating wrt t gives

$$\sigma_{11} = \mu \cos(\dot{\gamma}_0 t) - \mu \tag{30}$$
$$\sigma_{22} = -\mu \cos(\dot{\gamma}_0 t) + \mu, \tag{31}$$

where the constants from the integration have already been adopted to the initial conditions. Here it becomes evident that an arbitrary mixing of material law and co-rotational rate can result in unreasonable stress-strain relations when the strains become large.

For the polar rate, we need an expression for $\boldsymbol{R}$. From the obvious time-independent eigendirection $\boldsymbol{e}_3$ to the eigenvalue 1 it follows that $\boldsymbol{R}$ has the form

$$R_{ik} = \begin{bmatrix} c & s & 0 \\ -s & c & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{32}$$

with the abbreviations $c = \cos(\phi)$ and $s = \sin(\phi)$. One can tell that

$$R_{ik} = \begin{bmatrix} -s\dot{\phi} & c\dot{\phi} & 0 \\ -c\dot{\phi} & -s\dot{\phi} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{33}$$

and determine $\dot{\boldsymbol{R}}\boldsymbol{R}^T$ to

$$\dot{\boldsymbol{R}}\boldsymbol{R}^T = \dot{\phi}(\boldsymbol{e}_1 \otimes \boldsymbol{e}_2 - \boldsymbol{e}_2 \otimes \boldsymbol{e}_1). \tag{34}$$

We see that we get a similar result as for the Zaremba-Jaumann rate, since $\dot{\boldsymbol{R}}\boldsymbol{R}^T$ is like $\boldsymbol{W}$ anti-symmetric, but with $\dot{\phi}$ instead of $\dot{\gamma}_0/2$,

$$\dot{\sigma}_{11} = -2\dot{\phi}\sigma_{12} \tag{35}$$
$$\dot{\sigma}_{12} = \dot{\gamma}_0\mu + \dot{\phi}(\sigma_{11} - \sigma_{22}) \tag{36}$$
$$\dot{\sigma}_{22} = 2\dot{\phi}\sigma_{12} \tag{37}$$
$$\dot{\sigma}_{13} = -\dot{\phi}\sigma_{23} \tag{38}$$
$$\dot{\sigma}_{23} = \dot{\phi}\sigma_{13} \tag{39}$$
$$\dot{\sigma}_{33} = 0. \tag{40}$$

For σ_{13} and σ_{23} the solution is equivalent to the results for the Jaumann rate, i.e., we have a zero oscillation. To resolve the rest we need to determine $\dot{\phi}$. A straightforward but tedious way to calculate the polar decomposition is to calculate $\boldsymbol{C} = \boldsymbol{F}^T\boldsymbol{F} = \boldsymbol{U}^2$, determine the square root of $\boldsymbol{C}$ to access $\boldsymbol{U}$ by determining the spectral decomposition of $\boldsymbol{C}$, invert $\boldsymbol{U}$ and finally get $\boldsymbol{R} = \boldsymbol{F}\boldsymbol{U}^{-1}$. One can also attempt to solve the non-linear system of equations directly, which is in our case

$$F_{ij} = R_{ik}U_{kj} = \begin{bmatrix} c & s & 0 \\ -s & c & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} & 0 \\ U_{12} & U_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{41}$$

We can identify the four equations

$$1 = cU_{11} + sU_{12} \tag{42}$$
$$\gamma = cU_{12} + sU_{22} \tag{43}$$
$$0 = -sU_{11} + cU_{12} \tag{44}$$
$$1 = -sU_{12} + cU_{22} \tag{45}$$

By linear combinations of these equations with the factors c and s and the use of $s^2 + c^2 = 1$ we can get explicit expressions for the U_{ij}, namely s(43)+c(45), c(43)-s(45) and c(42)-s(44) give

$$U_{22} = s\gamma + c \tag{46}$$
$$U_{12} = c\gamma - s \tag{47}$$
$$U_{11} = c \tag{48}$$

We still need to find ϕ. Inserting U_{11} and U_{12} into eq. (42) gives

$$s^2 = -s^2 + sc\gamma, \tag{49}$$

where $1 = c^2 + s^2$ has been used to eliminate c^2. Rearranging for γ gives

$$\gamma = 2\tan(\phi). \tag{50}$$

We can now give $\dot{\phi}$,

$$\dot{\phi} = \frac{2\dot{\gamma}_0}{4 + \dot{\gamma}_0^2 t^2}, \tag{51}$$

leading to the following system of ODE for σ_{12}, σ_{11} and σ_{22}:

$$\dot{\sigma}_{11} = -\frac{4\dot{\gamma}_0}{4 + \dot{\gamma}_0^2 t^2}\sigma_{12} \tag{52}$$

$$\dot{\sigma}_{12} = \dot{\gamma}_0\mu + \frac{2\dot{\gamma}_0}{4 + \dot{\gamma}_0^2 t^2}(\sigma_{11} - \sigma_{22}) \tag{53}$$

$$\dot{\sigma}_{22} = \frac{4\dot{\gamma}_0}{4 + \dot{\gamma}_0^2 t^2}\sigma_{12} \tag{54}$$

With the initial conditions $\sigma_{11}(0) = \sigma_{22}(0) = 0$ we can conclude that $\sigma_{11}(t) = -\sigma_{22}(t)$, leaving two coupled equations,

$$\dot{\sigma}_{11} = -\frac{4\dot{\gamma}_0}{4 + \dot{\gamma}_0^2 t^2}\sigma_{12} \tag{55}$$

$$\dot{\sigma}_{12} = \dot{\gamma}_0\mu + \frac{4\dot{\gamma}_0}{4 + \dot{\gamma}_0^2 t^2}\sigma_{11}. \tag{56}$$

The latter system can be rewritten as

$$\begin{bmatrix}\dot{\sigma}_{11}\\ \dot{\sigma}_{12}\end{bmatrix} = \kappa \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix} \begin{bmatrix}\sigma_{11}\\ \sigma_{12}\end{bmatrix} + \begin{bmatrix}0\\ \mu\dot{\gamma}_0\end{bmatrix}, \qquad \kappa = \frac{4\dot{\gamma}_0}{4 + \dot{\gamma}_0^2 t^2}. \tag{57}$$

We can decouple the two equations by considering the right eigenvector problem, which gives the eigenvectors $[i, 1]/\sqrt{2}$ and $[-i, 1]/\sqrt{2}$. Putting these column-wise into a matrix results in a transformation matrix S_{ij},

$$S_{ij} = \frac{1}{\sqrt{2}}\begin{bmatrix}i & -i\\ 1 & 1\end{bmatrix}. \tag{58}$$

We can now by

$$\begin{bmatrix}\sigma_{11}\\ \sigma_{12}\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}i & -i\\ 1 & 1\end{bmatrix}\begin{bmatrix}a\\ b\end{bmatrix}, \tag{59}$$

replace the $\overset{(\cdot)}{\sigma}_{xy}$ by $\overset{(\cdot)}{a}$, $\overset{(\cdot)}{b}$, resulting in

$$\begin{bmatrix}\dot{a}\\ \dot{b}\end{bmatrix} = \frac{\kappa}{2}\begin{bmatrix}-i & 1\\ i & 1\end{bmatrix}\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\begin{bmatrix}i & -i\\ 1 & 1\end{bmatrix}\begin{bmatrix}a\\ b\end{bmatrix} + \frac{1}{\sqrt{2}}\begin{bmatrix}-i & 1\\ i & 1\end{bmatrix}\begin{bmatrix}0\\ \mu\dot{\gamma}_0\end{bmatrix} \tag{60}$$

$$= \kappa\begin{bmatrix}i & 0\\ 0 & -i\end{bmatrix}\begin{bmatrix}a\\ b\end{bmatrix} + \frac{1}{\sqrt{2}}\begin{bmatrix}\mu\dot{\gamma}_0\\ \mu\dot{\gamma}_0\end{bmatrix}, \tag{61}$$

where the inverse S_{ij}^{-1} is needed. The decoupled ODE are

$$\dot{a} = \kappa i a + \mu\dot{\gamma}_0/\sqrt{2} \tag{62}$$

$$\dot{b} = -\kappa i b + \mu\dot{\gamma}_0/\sqrt{2}. \tag{63}$$

The general solution of an ODE of the form $y' = A(x) + B(x)y$ is $y = P_1 + P_2$, with $P_1 = P_0 \int \exp(B)\mathrm{d}x$, $P_2 = P_1/P_0 \int AP_0/P_1 \mathrm{d}x$. Here, P_0 is a constant that is adopted to the initial conditions, i.e., the integration is carried out without introducing integration constants. With $a(0) = 0$ and $b(0) = 0$ one obtains

$$a = e^{2i\mathrm{arctan}\left[\frac{\dot\gamma_0 t}{2}\right]}\mu\left(-\dot\gamma_0 t + 4\mathrm{arctan}\left[\frac{\dot\gamma_0 t}{2}\right] - 2i\mathrm{ln}\left[1+\frac{\dot\gamma_0^2 t^2}{4}\right]\right)/\sqrt{2} \tag{64}$$

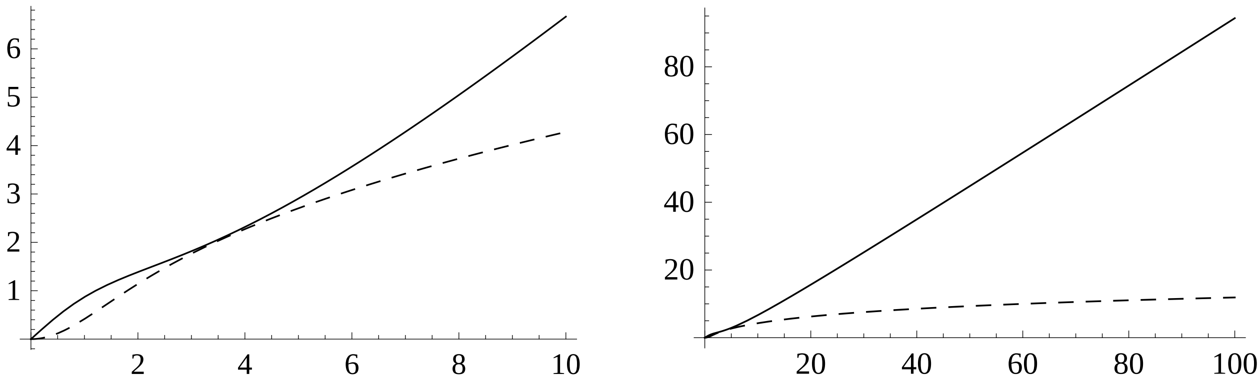
$$b = e^{-2i\mathrm{arctan}\left[\frac{\dot\gamma_0 t}{2}\right]}\mu\left(-\dot\gamma_0 t + 4\mathrm{arctan}\left[\frac{\dot\gamma_0 t}{2}\right] + 2i\mathrm{ln}\left[1+\frac{\dot\gamma_0^2 t^2}{4}\right]\right)/\sqrt{2}, \tag{65}$$

which gives with the substitution employed above the solution for σ_{xy},

$$\sigma_{11} = \frac{4\dot\gamma_0\mu t\left(\dot\gamma_0 t - 4\mathrm{arctan}\left[\frac{\dot\gamma_0 t}{2}\right]\right) - 2\mu\left(-4+\dot\gamma_0^2 t^2\right)\ln\left[1+\frac{\dot\gamma_0^2 t^2}{4}\right]}{4+\dot\gamma_0^2 t^2} \tag{66}$$

$$\sigma_{12} = \frac{\mu\left(-4+\dot\gamma_0^2 t^2\right)\left(\dot\gamma_0 t - 4\mathrm{arctan}\left[\frac{\dot\gamma_0 t}{2}\right]\right) + 8\dot\gamma_0\mu t\mathrm{ln}\left[1+\frac{\dot\gamma_0^2 t^2}{4}\right]}{4+\dot\gamma_0^2 t^2}. \tag{67}$$

The imaginary unit i must cancel out, since our initial system is real. In order to get an impression of the behaviour of such a material, a plot of σ_{12} (solid line) and $\sigma_{22} = -\sigma_{11}$ (dashed line) in MPa over t in s is given, for $\mu = 1\text{MPa}$, $\dot\gamma_0 = 1\text{s}^{-1}$.



The stress components σ_{12}, σ_{11} and σ_{22} tend to infinity, σ_{12} linearly and σ_{11} and σ_{22} logarithmically. One can argue that this is a decent material behaviour, at least in a shear test with zero initial stresses. However, to be sure that one uses meaningful rate-formulation of an elastic law, one should always start with its total form (ideally obtained from a strain energy), and take its material time derivative, see Section 2.2.

2 St.-Venant Kirchhoff law

2.1 Stress-Strain-response in characteristic tests

Task: Consider

- **(1)** a purely dilatorical deformation
- **(2)** a uniaxial tension test in $\boldsymbol{e}_1$-direction without lateral straining
- **(3)** a uniaxial tension test

for the isotropic St.-Venant-Kirchhoff material. Assume that $\boldsymbol{F}$ has the form $\boldsymbol{F} = \lambda_1 \boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + \lambda_2 \boldsymbol{e}_2 \otimes \boldsymbol{e}_2 + \lambda_3 \boldsymbol{e}_3 \otimes \boldsymbol{e}_3$. Discuss the function $\sigma_{11}(\lambda_1)$ in all three cases for $0 < \lambda_1$.

Solution: The Cauchy stresses are obtained by

$$\boldsymbol{\sigma} = \frac{1}{J}\boldsymbol{F}\boldsymbol{T}\boldsymbol{F}^T, \qquad J = \det\boldsymbol{F} = \lambda_1\lambda_2\lambda_3 \tag{68}$$

from the second Piola-Kirchhoff-stress $\boldsymbol{T}$. These are, following the St-Venant-Kirchhoff, law coupled to Greens strain tensor $\boldsymbol{E}$,

$$\boldsymbol{T} = \mathbb{C}[\boldsymbol{E}] \tag{69}$$

$$\boldsymbol{E} = {}^1\!/\!_2(\boldsymbol{F}^T\boldsymbol{F} - \boldsymbol{I}). \tag{70}$$

In the case of isotropy it is convenient to use the representation with Lamés constants

$$\boldsymbol{T} = \lambda \mathrm{tr}(\boldsymbol{E})\boldsymbol{I} + 2\mu\boldsymbol{E}. \tag{71}$$

With this equations, we may examine the three cases:

(1) All stretching is equal to λ_1, i.e. $\boldsymbol{F} = \lambda_1 \boldsymbol{I}$. Inserting this $\boldsymbol{F}$ into the latter equations gives $\boldsymbol{\sigma} = ({}^3\!/\!_2\lambda - \mu)(\lambda_1 - {}^1\!/\!_{\lambda_1})$, thus

$$\sigma_{11} = ({}^3\!/\!_2\lambda - \mu)(\lambda_1 - {}^1\!/\!_{\lambda_1}). \tag{72}$$

In the interval $0 < \lambda_1$, this function is monotonically growing. It has pole at $\lambda_1 = 0$, coming from $-\infty$, passes its root $\sigma_{11} = 0$ at $\lambda_1 = 1$ and approaches asymptotically the linear function $\sigma_\infty = ({}^3\!/\!_2\lambda - \mu)\lambda_1$. At $\lambda_1 = 1$, the slope is $3\lambda - 2\mu = 3K$ (K is the compression modulus).

(2) The lateral stretching is fixed to $\lambda_2 = \lambda_3 = 1$. Inserting this gives $\boldsymbol{E} = {}^1\!/\!_2(\lambda_1^2 - 1)\boldsymbol{e}_1 \otimes \boldsymbol{e}_1$. The second Piola-Kirchoff stresses are then $\boldsymbol{T} = {}^1\!/\!_2(\lambda(\lambda_1^2 - 1) + 2\mu(\lambda_1^2 - 1))\boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + {}^1\!/\!_2\lambda(\lambda_1^2 - 1)(\boldsymbol{e}_2 \otimes \boldsymbol{e}_2 + \boldsymbol{e}_3 \otimes \boldsymbol{e}_3)$. The σ_{11}-component results in $\sigma_{11} = ({}^\lambda\!/\!_2 + \mu)(\lambda_1^3 - \lambda_1)$. This antimetric cubic polynom has roots at $\lambda_1 = \{-1, 0, 1\}$. Thus, in the considered interval, it starts with a negative slope at $(0,0)$, has a minimum at $({}^1\!/\!_{\sqrt{3}}, {}^{-2({}^\lambda\!/\!_2+\mu)}\!/\!_{3^{3/2}})$, followed by a root at $(1,0)$ and then tending to $+\infty$. One would consider such material behaviour as unrealistic, since in a compression test the stress drops to zero as the body is compressed to zero thickness.

(3) This case is a little bit more complicated, since the lateral stretching is unknown. It must be determined from the condition that the lateral stress is zero in an ordinary uniaxial tension test. However, it is due to the isotropy the same in $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$ direction. Thus we take $\boldsymbol{F} = \lambda_1\boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + \lambda_2(\boldsymbol{e}_2 \otimes \boldsymbol{e}_2 + \boldsymbol{e}_3 \otimes \boldsymbol{e}_3)$ and calculate the Cauchy stresses,

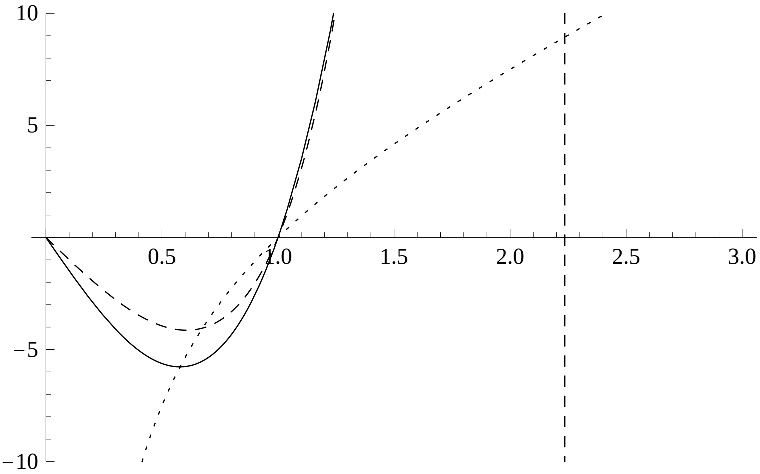
$$\sigma_{11} = \frac{\lambda_1}{\lambda_2^2}\left(\left(\frac{\lambda}{2} + \mu\right)(\lambda_1^2 - 1) + \lambda(\lambda_2^2 - 1)\right) \tag{73}$$

$$\sigma_{22} = \frac{1}{\lambda_1}\left((\lambda + \mu)(\lambda_2^2 - 1) + \frac{\lambda}{2}(\lambda_1^2 - 1)\right) \tag{74}$$

From $\sigma_{22} = 0$ we can determine λ_2^2, which can be substituted in the equation for σ_{11}, giving

$$\sigma_{11} = \frac{(\lambda_1^3 - \lambda_1)\mu(3\lambda + 2\mu)}{\lambda(\lambda_1^2 - 3) - 2\mu}. \tag{75}$$

This function has its roots at $\{-1, 0, 1\}$ and poles at $\pm\sqrt{2\mu/\lambda + 3}$. It starts at $(0, 0)$ with a negative slope, passes a local minimum, passes its root at $(1, 0)$, and goes to $+\infty$ as the pole at $\sqrt{2\mu/\lambda + 3}$ is approached. Passing the pole it comes from $-\infty$ and approaches the linear function $-\lambda_1(3\mu + 2\mu^2/\lambda)$. Having a look at the graph (dashed line) indicates that such a stress-stretch-response is clearly not expected in an ordinary tension test. Thus, St.-Venant-Kirchhoff elasticity is only feasible for relatively small strains. The question for valid large strain elastic laws arises, which is the topic of another class.



The cases (1) (dotted line), (2) (solid line) and (3) (dashed line) for $\lambda = \mu = 1\mathrm{MPa}$

2.2 Incremental form of the St.-Venant-Kirchhoff law

The starting point is again

$$\boldsymbol{\sigma} = \frac{1}{J}\boldsymbol{F}\boldsymbol{T}\boldsymbol{F}^T, \qquad \boldsymbol{T} = \frac{1}{2}\mathbb{C}_0 \cdot\cdot (\boldsymbol{F}^T\boldsymbol{F} - \boldsymbol{I}), \tag{76}$$

with the Cauchy stresses $\boldsymbol{\sigma}$, the second Piola-Kirchhoff stresses $\boldsymbol{T}$, the deformation gradient $\boldsymbol{F}$ and its determinant J.
Task: Derive from the total constitutive material law $\boldsymbol{\sigma}(\boldsymbol{F})$ the rate-form constitutive law of the form $\dot{\boldsymbol{\sigma}}(\boldsymbol{L})$.
Solution: The material time derivative is defined as $\frac{\partial \cdot}{\partial t}\big|_{\boldsymbol{X}=\text{const.}}$. The material time derivative of $\boldsymbol{\sigma}$ is

$$\dot{\boldsymbol{\sigma}} = -\frac{\dot{J}}{J^2}\boldsymbol{F}\boldsymbol{T}\boldsymbol{F}^T + \frac{1}{J}(\dot{\boldsymbol{F}}\boldsymbol{T}\boldsymbol{F}^T + \boldsymbol{F}\dot{\boldsymbol{T}}\boldsymbol{F}^T + \boldsymbol{F}\boldsymbol{T}\dot{\boldsymbol{F}}^T), \tag{77}$$

i.e. four times the product rule. With

$$\dot{\boldsymbol{F}} = \boldsymbol{L}\boldsymbol{F} \tag{78}$$

$$\dot{J} = \partial\det(\boldsymbol{F})/\partial\boldsymbol{F} \cdot\cdot \dot{\boldsymbol{F}} = J\boldsymbol{L} \cdot\cdot \boldsymbol{I} = J\mathrm{tr}(\boldsymbol{L}) \tag{79}$$

$\boldsymbol{\sigma}$ is simplified to

$$\dot{\boldsymbol{\sigma}} = -\mathrm{tr}(\boldsymbol{L})\boldsymbol{\sigma} + \boldsymbol{L}\boldsymbol{\sigma} + \boldsymbol{\sigma}\boldsymbol{L}^T + \frac{1}{J}\boldsymbol{F}\dot{\boldsymbol{T}}\boldsymbol{F}^T. \tag{80}$$

Moreover,

$$\dot{\boldsymbol{T}} = \frac{1}{2}\mathbb{C}_0 \cdot\cdot \dot{\boldsymbol{C}} = \mathbb{C}_0 \cdot\cdot (\boldsymbol{F}^T\boldsymbol{D}\boldsymbol{F}), \tag{81}$$

which is due to

$$\dot{\boldsymbol{C}} = 2\boldsymbol{F}^T\boldsymbol{D}\boldsymbol{F}. \tag{82}$$

Inserting this we find

$$\dot{\boldsymbol{\sigma}} = -\mathrm{tr}(\boldsymbol{L})\boldsymbol{\sigma} + \boldsymbol{L}\boldsymbol{\sigma} + \boldsymbol{\sigma}\boldsymbol{L}^T + \frac{1}{J}(\boldsymbol{F} * \mathbb{C}_0) \cdot\cdot \boldsymbol{D}. \tag{83}$$

To get rid of the last reference to the reference placement we have to eliminate $\boldsymbol{F}$. Thus we introduce the substitution $\mathbb{C} = \boldsymbol{F} * \mathbb{C}_0$. The time derivative of $\mathbb{C}$ is then

$$\dot{\mathbb{C}} = (L_{ix}C_{xjkl} + L_{jx}C_{ixkl} + L_{kx}C_{ijxl} + L_{lx}C_{ijkx})\boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \otimes \boldsymbol{e}_l = \boldsymbol{L} \boxtimes \mathbb{C}. \tag{84}$$

Thus, we have transformed the total constitutive material law into a rate form material law,

$$\dot{\boldsymbol{\sigma}} = -\mathrm{tr}(\boldsymbol{L})\boldsymbol{\sigma} + \boldsymbol{L} \boxtimes \boldsymbol{\sigma} + \frac{1}{J}\mathbb{C} \cdot\cdot \boldsymbol{D}, \tag{85}$$

$$\dot{\mathbb{C}} = \boldsymbol{L} \boxtimes \mathbb{C}, \tag{86}$$

$$\dot{J} = J\mathrm{tr}(\boldsymbol{L}). \tag{87}$$

A test of plausibility is for example to assume $\boldsymbol{L} \in Skw$, which leads to $\boldsymbol{F} \in Orth^+$. Then, J remains constant and $\boldsymbol{D} = \boldsymbol{0}$, and $\boldsymbol{\sigma}$ and $\mathbb{C}_0$ are merely rotated by

$$\dot{\boldsymbol{\sigma}} = \boldsymbol{W} \boxtimes \boldsymbol{\sigma} \qquad \rightarrow \boldsymbol{\sigma}(t) = \boldsymbol{Q}(t) * \boldsymbol{\sigma}(t_0) \tag{88}$$

$$\dot{\mathbb{C}} = \boldsymbol{W} \boxtimes \mathbb{C} \qquad \rightarrow \mathbb{C}(t) = \boldsymbol{Q}(t) * \mathbb{C}(t_0) \tag{89}$$

We have derived a rate form material law from a total form material law by time differentiation. Thus, the integrability and hence the existence of the strain energy, i.e., hyperelasticity is ensured, since our total form material law is obtained from an elastic energy. One can see that the arbitrariness explored in Section 1 is eleminated: we found that we have to employ the Truesdell rate (eq. 85), in conjunction with a specific evolution of $\mathbb{C}$. One must be careful when using rate form (hypoelastic) elastic laws.

3 Deriving the elastic law from the elastic energy

Task: Determine the stress-strain relations in terms of $\boldsymbol{\sigma}(\boldsymbol{B})$ **(1)** for the Ciarlet model (compressible, with Lamé's constants λ and μ),

$$w_{\text{Ciarlet}} = \frac{\lambda}{4}(\mathrm{III}_{\boldsymbol{B}} - 1) - (\frac{\lambda}{4} + \frac{\mu}{2})\ln(\mathrm{III}_{\boldsymbol{B}}) + \frac{\mu}{2}(\mathrm{I}_{\boldsymbol{B}} - 3) \tag{90}$$

$$\tag{91}$$

and **(2)** the Mooney-Rivlin model (incompressible)

$$w_{\text{MR}} = \frac{\alpha}{2}(\mathrm{I}_{\boldsymbol{B}} - 3) + \frac{\beta}{2}(\mathrm{II}_{\boldsymbol{B}} - 3). \tag{92}$$

Comment: The isotropic elastic energies may be written as functions of the invariants and/or the eigenvalues of any strain or stretch tensor, which can be transformed into one another. The advantages of the invariant representation are

- invariants can easily be replaced by the eigenvalues by
  $\mathrm{I} = \lambda_1 + \lambda_1 + \lambda_1, \mathrm{II} = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1, \mathrm{III} = \lambda_1\lambda_2\lambda_3$
- taking the derivative with respect to the tensor gives combinations of invariants and powers of the tensor, thus one can stick to a fully symbolic notation without choosing a specific basis
- the invariants are obtained easily from the components of the tensor
- the determinant of a stretch tensor is a measure for the volume change (which can be very helpful).

The advantages of the eigenvalue representation are

- the strain energy is symmetric with respect to the interchanging of the arguments
- the eigenvalues require a similar treatment, e.g., the derivatives with respect to the eigenvalues are obtained similarly

Solution: **(1)** For hyperelastic materials without internal constraints, the Cauchy stresses are obtained by

$$\boldsymbol{\sigma} = 2\rho\frac{\mathrm{d}w}{\mathrm{d}\boldsymbol{B}}\boldsymbol{B}. \tag{93}$$

The derivative with respect to $\boldsymbol{B}$ is given by

$$\frac{\mathrm{d}w}{\mathrm{d}\boldsymbol{B}} = \frac{\partial w}{\partial \mathrm{I}_{\boldsymbol{B}}}\frac{\partial \mathrm{I}_{\boldsymbol{B}}}{\partial \boldsymbol{B}} + \frac{\partial w}{\partial \mathrm{II}_{\boldsymbol{B}}}\frac{\partial \mathrm{II}_{\boldsymbol{B}}}{\partial \boldsymbol{B}} + \frac{\partial w}{\partial \mathrm{III}_{\boldsymbol{B}}}\frac{\partial \mathrm{III}_{\boldsymbol{B}}}{\partial \boldsymbol{B}} \tag{94}$$

in terms of invariants or by

$$\frac{\mathrm{d}w}{\mathrm{d}\boldsymbol{B}} = \sum_{i=1}^{3}\frac{\partial w}{\partial \lambda_i}\boldsymbol{P}_i \tag{95}$$

in terms of the Eigenvalues of $\boldsymbol{B}$, with the Eigenprojectors $\boldsymbol{P}_i$. The derivatives of the invariants with respect to a tensor are

$$\frac{\partial \mathrm{I}_{\boldsymbol{B}}}{\partial \boldsymbol{B}} = \boldsymbol{I}, \qquad \frac{\partial \mathrm{II}_{\boldsymbol{B}}}{\partial \boldsymbol{B}} = \mathrm{I}_{\boldsymbol{B}}\boldsymbol{I} - \boldsymbol{B}^T, \qquad \frac{\partial \mathrm{III}_{\boldsymbol{B}}}{\partial \boldsymbol{B}} = \mathrm{III}_{\boldsymbol{B}}\boldsymbol{B}^{-T}. \tag{96}$$

Thus, using the invariant representation, only the derivatives of w with respect to the invariants are missing. For Ciarlets strain energy we obtain

$$\frac{\mathrm{d}w_{\text{Ciarlet}}}{\mathrm{d}\boldsymbol{B}} = \frac{\mu}{2}\boldsymbol{I} + \left(\frac{\lambda}{4} - \frac{1}{\mathrm{III}_{\boldsymbol{B}}}(\frac{\lambda}{4} + \frac{\mu}{2})\right)\mathrm{III}_{\boldsymbol{B}}\boldsymbol{B}^{-1}. \tag{97}$$

Transforming to the Cauchy stress and reordering gives

$$\boldsymbol{\sigma} = \rho(\mu(\boldsymbol{B} - \boldsymbol{I}) + \frac{\lambda}{2}(\mathrm{III}_{\boldsymbol{B}} - 1)\boldsymbol{I}) \tag{98}$$

The result is plausible, since the deformation-free state $\boldsymbol{B} = \boldsymbol{I}$ results in zero stresses. Using the eigenvalues, we firstly represent w_{Ciarlet} by

$$w_{\text{Ciarlet}} = \frac{\lambda}{4}(\lambda_1\lambda_2\lambda_3 - 1) - (\frac{\lambda}{4} + \frac{\mu}{2})\ln(\lambda_1\lambda_2\lambda_3) + \frac{\mu}{2}(\lambda_1 + \lambda_2 + \lambda_3 - 3) \tag{99}$$

The derivative with respect to $\boldsymbol{B}$ is obtained by

$$\frac{\mathrm{d}w_{\text{Ciarlet}}}{\mathrm{d}\boldsymbol{B}} = \sum_{i=1}^{3}\left(\frac{\lambda}{4}\lambda_{i+1}\lambda_{i+2} - \frac{\lambda_{i+1}\lambda_{i+2}}{\lambda_1\lambda_2\lambda_3}(\frac{\lambda}{4} + \frac{\mu}{2}) + \frac{\mu}{2}\right)\boldsymbol{P}_i. \tag{100}$$

The first term in the sum can be expanded by λ_i such that the determinant can be pulled out of the sum, in the second term the fraction simplifies to $1/\lambda_i$:

$$\frac{\mathrm{d}w_{\text{Ciarlet}}}{\mathrm{d}\boldsymbol{B}} = \frac{\lambda\mathrm{III}_{\boldsymbol{B}}}{4}\sum_{i=1}^{3}\frac{1}{\lambda_i}\boldsymbol{P}_i - (\frac{\lambda}{4} + \frac{\mu}{2})\sum_{i=1}^{3}\frac{1}{\lambda_i}\boldsymbol{P}_i + \frac{\mu}{2}\sum_{i=1}^{3}\boldsymbol{P}_i \tag{101}$$

One sees that the first two sums correspond to $\boldsymbol{B}^{-1}$ and the third to $\boldsymbol{I}$, which allows to summarize to eq. (97).
(2) For materials with internal constraints we have a decomposition of the stress tensor into a constitutive and a reaction part. Our material law gives only results for the constitutive part. For incompressible isotropic materials we have

$$\boldsymbol{\sigma} = -p\boldsymbol{I} + 2\rho(\frac{\partial w}{\partial \mathrm{I}_{\boldsymbol{B}}}\boldsymbol{B} - \frac{\partial w}{\partial \mathrm{II}_{\boldsymbol{B}}}\boldsymbol{B}^{-1}), \tag{102}$$

where the pressure part p corresponds to the reaction stresses. Also it should be noted that in eq. (94) the last term is zero since the movement of the body is isochoric, thus $\partial\mathrm{III}_{\boldsymbol{B}}/\partial\boldsymbol{B}$ is always zero. Therefore, any dependence on III is blanked out, and if w is given in terms of invariants, $\mathrm{III}_{\boldsymbol{B}}$ should not appear. Using the latter equation we obtain easily the Cauchy stresses,

$$\boldsymbol{\sigma} = -p\boldsymbol{I} + \rho(\alpha\boldsymbol{B} - \beta\boldsymbol{B}^{-1}). \tag{103}$$

It should be noted that, although the constitutive part contains a spherical contribution $\kappa\boldsymbol{I}$, this is overridden anyway by the reaction part. Thus, we can decompose further into the purely dilatoric (reaction) and deviatoric (constitutive) part:

$$\boldsymbol{\sigma} = -p^*\boldsymbol{I} + \rho(\alpha\boldsymbol{B} - \beta\boldsymbol{B}^{-1})'. \tag{104}$$

Here we see again that the constitutive part vanishes in case that $\boldsymbol{B} = \boldsymbol{I}$.

Homework: **(1)** Assuming small strains, the constitutive part of the Mooney-Rivlin-law can be approximated by $\boldsymbol{\sigma} = -p^*\boldsymbol{I} + 2G\boldsymbol{E}'$. For that case, relate the coefficients α and β to G.
(2) The St.-Venant-Kirchhoff law is $\boldsymbol{T} = \lambda\mathrm{I}_{\boldsymbol{E}}\boldsymbol{I} + 2\mu\boldsymbol{E}$ (second Piola-Kirchhoff stresses and Greens strain). Determine the corresponding strain energy $w(\boldsymbol{E})$ which gives $\partial w/\partial\boldsymbol{E} = \boldsymbol{T}$ in terms of invariants of $\boldsymbol{E}$, eigenvalues of $\boldsymbol{E}$ and invariants of $\boldsymbol{U}$.

4 Rubber Balloon

Task: A hollow Sphere made from incompressible Mooney-Rivlin-material is subjected to internal and external hydrostatic pressure. Determine the resulting stresses and deformations.
We use spherical coordinates,

$$x = r\cos\theta\cos\phi \tag{105}$$
$$y = r\cos\theta\sin\phi \tag{106}$$
$$z = r\sin\theta, \tag{107}$$

with $-\pi/2 < \theta < \pi/2$, $-\pi < \phi < \pi$, $0 < r$. A location vector is given through

$$\boldsymbol{x} = x_i\boldsymbol{e}_i = r(\cos\theta\cos\phi\boldsymbol{e}_x + \cos\theta\sin\phi\boldsymbol{e}_y + \sin\theta\boldsymbol{e}_z). \tag{108}$$

The tangential spherical basis is obtained by

$$\boldsymbol{e}_r = \boldsymbol{x}_{,r} = \cos\theta\cos\phi\boldsymbol{e}_x + \cos\theta\sin\phi\boldsymbol{e}_y + \sin\theta\boldsymbol{e}_z \tag{109}$$
$$\boldsymbol{e}_\phi = \boldsymbol{x}_{,\phi} = r(-\cos\theta\sin\phi\boldsymbol{e}_x + \cos\theta\cos\phi\boldsymbol{e}_y) \tag{110}$$
$$\boldsymbol{e}_\theta = \boldsymbol{x}_{,\theta} = r(-\sin\theta\cos\phi\boldsymbol{e}_x - \sin\theta\sin\phi\boldsymbol{e}_y + \cos\theta\boldsymbol{e}_z). \tag{111}$$

Normalising the tangent basis yields

$$\boldsymbol{e}_r^* = \boldsymbol{e}_r \tag{112}$$
$$\boldsymbol{e}_\phi^* = \frac{1}{r\cos\theta}\boldsymbol{e}_\phi = -\sin\phi\boldsymbol{e}_x + \cos\phi\boldsymbol{e}_y \tag{113}$$
$$\boldsymbol{e}_\theta^* = \frac{1}{r}\boldsymbol{e}_\theta = -\sin\theta\cos\phi\boldsymbol{e}_x - \sin\theta\sin\phi\boldsymbol{e}_y + \cos\theta\boldsymbol{e}_z. \tag{114}$$

We go on using, after dropping the upper $*$ index, the normalised basis. We will need derivatives of the base vectors with respect to the coordinates, which is given exemplarily for $\boldsymbol{e}_r$:

$$\boldsymbol{e}_{r,r} = \boldsymbol{0} \tag{115}$$
$$\boldsymbol{e}_{r,\phi} = -\cos\theta\sin\phi\boldsymbol{e}_x + \cos\theta\cos\phi\boldsymbol{e}_y \tag{116}$$
$$\boldsymbol{e}_{r,\theta} = -\sin\theta\cos\phi\boldsymbol{e}_x - \sin\theta\sin\phi\boldsymbol{e}_y - \sin\theta\boldsymbol{e}_z \tag{117}$$

To obtain rules how to calculate in the tangent basis, we need to represent the latter vectors in the tangent basis. One finds

$$\boldsymbol{e}_{r,r} = \boldsymbol{0} \tag{118}$$
$$\boldsymbol{e}_{r,\phi} = \cos\theta\boldsymbol{e}_\phi \tag{119}$$
$$\boldsymbol{e}_{r,\theta} = \boldsymbol{e}_\theta \tag{120}$$

The same is done for $\boldsymbol{e}_\theta$ and $\boldsymbol{e}_\phi$,

$$\boldsymbol{e}_{\phi,r} = \boldsymbol{0} \tag{121}$$
$$\boldsymbol{e}_{\phi,\phi} = -\cos\theta\boldsymbol{e}_r + \sin\theta\boldsymbol{e}_\theta \tag{122}$$
$$\boldsymbol{e}_{\phi,\theta} = \boldsymbol{0} \tag{123}$$
$$\boldsymbol{e}_{\theta,r} = \boldsymbol{0} \tag{124}$$
$$\boldsymbol{e}_{\theta,\phi} = -\sin\theta\boldsymbol{e}_\phi \tag{125}$$
$$\boldsymbol{e}_{\theta,\theta} = -\boldsymbol{e}_r \tag{126}$$

Further, we will need the differential operator

$$\nabla = \frac{\partial}{\partial x}\boldsymbol{e}_x + \frac{\partial}{\partial y}\boldsymbol{e}_y + \frac{\partial}{\partial z}\boldsymbol{e}_z \tag{127}$$

with respect to the normalised tangent basis. Firstly, we may represent the $\boldsymbol{e}_{x,y,z}$ as a linear combination of $\boldsymbol{e}_{r,\phi,\theta}$. Secondly, we can replace the derivatives with respect the cartesian basis by expanding with the chain rule, for example for x,

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial}{\partial \phi}\frac{\partial \phi}{\partial x} + \frac{\partial}{\partial \theta}\frac{\partial \theta}{\partial x}, \tag{128}$$

where the derivatives $\frac{\partial(r,\phi,\theta)}{\partial x}$ are known from the coordinate change (eqs. 107). All this requires of course the bijectivity of the coordinate change, which is the case for the restrictions for r, ϕ and θ given above. The calculation is straight foreward, but lengthy, so only the result

$$\nabla = \frac{\partial}{\partial r}\boldsymbol{e}_r + \frac{1}{r}\frac{\partial}{\partial \theta}\boldsymbol{e}_\theta + \frac{1}{r\cos\theta}\frac{\partial}{\partial \phi}\boldsymbol{e}_\phi \tag{129}$$

is given. Next, we make an approach for the motion of the body. Assuming that only the radius changes, we write

$$r = \mu(R)R \tag{130}$$

$$\theta = \Theta \tag{131}$$

$$\phi = \Phi \tag{132}$$

It is not necessary, but convenient, to introduce the ratio $\mu = r(R)/R$, which is useful later on. Now we can give $\boldsymbol{F} = \boldsymbol{x} \otimes \nabla_{\boldsymbol{X}}$ ($\nabla_{\boldsymbol{X}}$ is the differential operator with respect to material coordinates, $\nabla_{\boldsymbol{x}}$ with respect to spatial coordinates),

$$\boldsymbol{F} = r(R)\boldsymbol{e}_r \otimes \left(\frac{\partial}{\partial R}\boldsymbol{e}_R + \frac{1}{R}\frac{\partial}{\partial \Theta}\boldsymbol{e}_\Theta + \frac{1}{R\cos\Theta}\frac{\partial}{\partial \Phi}\boldsymbol{e}_\Phi\right) \tag{133}$$

$$= r_{,R}\boldsymbol{e}_r \otimes \boldsymbol{e}_R + \frac{r}{R}\boldsymbol{e}_{r,\Theta} \otimes \boldsymbol{e}_\Theta + \frac{r}{R\cos\Phi}\boldsymbol{e}_{r,\Phi} \otimes \boldsymbol{e}_\Phi \tag{134}$$

Now since $\theta = \Theta$ and $\phi = \Phi$, we can replace all derivatives with respect to Θ and Φ by θ and ϕ. For the same reason, $\boldsymbol{e}_r = \boldsymbol{e}_R$ holds. Otherwise, we would have to expand by the chain rule, e.g. $\boldsymbol{e}_{r,\Theta} = \boldsymbol{e}_{r,\theta}\partial\theta/\partial\Theta$, using the motion of the body to determine the derivative $\partial\theta/\partial\Theta$. It remains

$$\boldsymbol{F} = r_{,R}\boldsymbol{e}_r \otimes \boldsymbol{e}_r + \frac{r}{R}(\boldsymbol{e}_\theta \otimes \boldsymbol{e}_\theta + \boldsymbol{e}_\phi \otimes \boldsymbol{e}_\phi). \tag{135}$$

The incompressibility condition imposes $\det \boldsymbol{F} = 1 = r_{,R}\mu^2$, which gives

$$r_{,R} = \mu^{-2}. \tag{136}$$

The resulting $\boldsymbol{F}$ is

$$\boldsymbol{F} = \mu^{-2}\boldsymbol{e}_r \otimes \boldsymbol{e}_r + \mu(\boldsymbol{e}_\theta \otimes \boldsymbol{e}_\theta + \boldsymbol{e}_\phi \otimes \boldsymbol{e}_\phi). \tag{137}$$

One form of the stress strain relation of the incompressible Mooney-Rivlin solid is

$$\boldsymbol{T} = a\boldsymbol{B} + b\boldsymbol{B}^{-1} - p(R)\boldsymbol{I}, \tag{138}$$

where $p(R)$ has to be determined from the local balance equation $\boldsymbol{T} \cdot \nabla_{\boldsymbol{x}} = \boldsymbol{0}$, and $\boldsymbol{B} = \boldsymbol{F}\boldsymbol{F}^T$. Note that $\boldsymbol{B}$ is completely determined by the movement of the body, which is given as a function of R, Θ and Φ. Therefore, the yet undetermined pressure should be introduced as a function of R (due to the spherical symmetry we drop the angles). For the stresses we obtain

$$\boldsymbol{T} = (a\mu^{-4} + b\mu^4 - p)\boldsymbol{e}_r \otimes \boldsymbol{e}_r + (a\mu^2 + b\mu^{-2} - p)(\boldsymbol{e}_\theta \otimes \boldsymbol{e}_\theta + \boldsymbol{e}_\phi \otimes \boldsymbol{e}_\phi). \tag{139}$$

The evaluation of $\boldsymbol{T} \cdot \nabla_{\boldsymbol{x}} = \boldsymbol{0}$ is quite lengthy, since we have to consider nine times a triple product rule. However, many derivatives vanish, and one finds

$$(T_{rr,r} + \frac{1}{r}(2T_{rr} - T_{\theta\theta} - T_{\phi\phi}))\boldsymbol{e}_r + \frac{T_{\phi\phi,\phi}}{r\cos\theta}\boldsymbol{e}_\phi + \frac{1}{r}(T_{\theta\theta,\theta} - \tan\theta(T_{\theta\theta} - T_{\phi\phi}))\boldsymbol{e}_\theta = \boldsymbol{0}, \tag{140}$$

with $T_{\phi\phi} = T_{\theta\theta}$. From this components, only the first one is not trivially equal to zero. Therefore, we have

$$T_{rr,r} + \frac{2}{r}(T_{rr} - T_{\theta\theta}) = 0 \tag{141}$$

which is the differential equation to be solved. Note that we have to take the derivative with respect to r. However, this must be replaced by the derivative with respect to R. We find with $r = \mu(R)R$ that

$$\frac{\partial\cdot}{\partial r} = \frac{\partial\cdot}{\partial R}\frac{\partial R}{\partial r} = \frac{\partial\cdot}{\partial R}\left(\frac{\partial r}{\partial R}\right)^{-1} = \frac{\partial\cdot}{\partial R}(r'(R))^{-1} \tag{142}$$

With the incompressibility condition, we had found that $r'(R) = \mu^{-2}$, i.e. the derivatives with respect to r are replaced by derivatives with respect to R times μ^2. Moreover, $r = \mu R$ is used. We obtain

$$T_{rr,R}\mu^2 + \frac{2}{\mu R}(T_{rr} - T_{\theta\theta}) = 0, \tag{143}$$

where

$$T_{rr,R} = -4a\mu^{-5}\mu' + 4b\mu^3\mu' - p'. \tag{144}$$

Here, the prime indicates always a derivative with respect to R. We replace μ' by

$$\mu(R) = \frac{r(R)}{R}, \qquad \mu' = \frac{r'}{R} - \frac{r}{R^2} = \frac{\mu^{-2}}{R} - \frac{\mu}{R}, \tag{145}$$

such that

$$T_{rr,R} = (\frac{\mu^{-2}}{R} - \frac{\mu}{R})(-4a\mu^{-5} + 4b\mu^3) - p'. \tag{146}$$

This all inserted into the remaining balance equation yields a differential equation for p,

$$(\frac{1}{R} - \frac{\mu^3}{R})(-4a\mu^{-5} + 4b\mu^3) - p'\mu^2 + \frac{2}{\mu R}(a\mu^{-4} + b\mu^4 - a\mu^2 - b\mu^{-2}) = 0. \tag{147}$$

Solving for p' gives

$$\frac{1}{R}\left(\mu^{-2}(1-\mu^3)(-4a\mu^{-5} + 4b\mu^3) + \frac{2}{\mu^3}(a\mu^{-4} + b\mu^4 - a\mu^2 - b\mu^{-2})\right) = p'. \tag{148}$$

This has to be integrated with respect to R. We have seen already that $\mu' = \mathrm{d}\mu/\mathrm{d}R = -\mu(1-\mu^{-3})/R$, which used for a subsitution of the domain of the integral,

$$p = \int -\left[\mu^{-2}(1-\mu^3)(-4a\mu^{-5} + 4b\mu^3) + 2(a\mu^{-7} + b\mu^1 - a\mu^{-1} - b\mu^{-5})\right]\frac{1}{\mu(1-\mu^{-3})}\mathrm{d}\mu \tag{149}$$

$$p = \int -\left[\mu^{-3}(1-\mu^3)(-4a\mu^{-5} + 4b\mu^3) + 2(a\mu^{-2}(\mu^{-6} - 1) + b(1-\mu^{-6}))\right]\frac{1}{(1-\mu^{-3})}\mathrm{d}\mu \tag{150}$$

Now one can replace $\mu^{-6} - 1 = -(1-\mu^{-3})(1+\mu^{-3})$ and $1 - \mu^3 = -\mu^3(1-\mu^{-3})$, which allows to cancel the denominator $1-\mu^{-3}$,

$$p = \int -\left[-\mu^{-3}\mu^3(-4a\mu^{-5} + 4b\mu^3) - 2a\mu^{-2}(1+\mu^{-3}) + 2b(1+\mu^{-3})\right]\mathrm{d}\mu \tag{151}$$

$$p = \int(-2a\mu^{-5} + 2a\mu^{-2} + 4b\mu^3 - 2b - 2b\mu^{-3})\mathrm{d}\mu \tag{152}$$

$$p = -2a\mu^{-1} + a\mu^{-4}/2 + b\mu^4 - 2b\mu + b\mu^{-2} + C. \tag{153}$$

With p, we can give T_{rr} as

$$T_{rr} = 2a\mu^{-1} + a\mu^{-4}/2 + 2b\mu - b\mu^{-2} - C_1. \tag{154}$$

Due to the incompressibility condition, we can express μ at any radius if μ is known at one specific radius. We have

$$\frac{\mathrm{d}r}{\mathrm{d}R} = \mu^{-2} = \frac{R^2}{r^2} \tag{155}$$

$$r^2 \mathrm{d}r = R^2 \mathrm{d}R \tag{156}$$

$$r^3 = R^3 + C_2 \tag{157}$$

$$C_2 = r^3 - R^3 \tag{158}$$

where the factor $1/3$ has been summarised in C_2. We now have two determine two constants C_1 and C_2 from the boundary conditions at the inner and outer radius. It is important to note that, due to the incompressibility, we can not impose the displacements on both boundaries at the same time. We take $T_{rr}(R_a) = 0$ and $T_{rr}(R_i) = -p_i$. We can take equation (154) at μ_i and μ_a and substract them to eliminate C_1,

$$-p_i = 2a(\mu_i^{-1} - \mu_a^{-1}) + a(\mu_i^{-4} - \mu_a^{-4})/2 + 2b(\mu_i - \mu_a) - b(\mu_i^{-2} - \mu_a^{-2}). \tag{159}$$

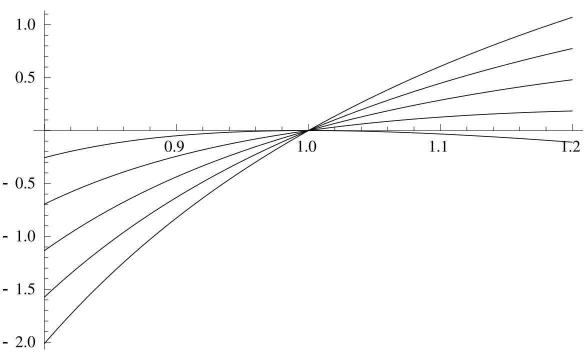
Next, we use eq. (158) to express μ_a as a function of μ_i,

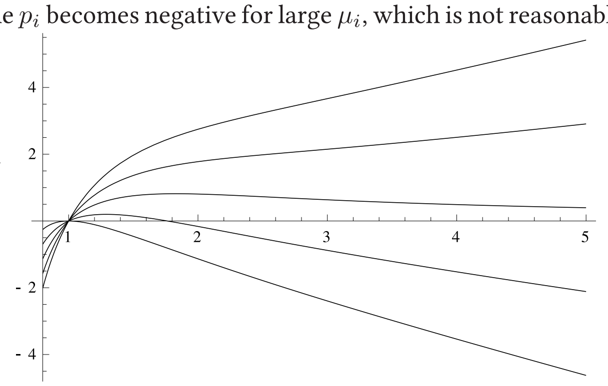
$$r_i^3 - R_i^3 = r_a^3 - R_a^3 \qquad \rightarrow \qquad \mu_a = \left(1 + \frac{R_i^3}{R_a^3}(\mu_i^3 - 1)\right)^{1/3}, \tag{160}$$

which can finally be used to give p_i as a function of μ_i,

$$p_i = b\left(\frac{1}{\mu_i^2} - 2\mu_i + \frac{R_a^3 + 2\left(-1+\mu_i^3\right)R_i^3}{R_a^3\left(\frac{R_a^3+\left(-1+\mu_i^3\right)R_i^3}{R_a^3}\right)^{2/3}}\right) + \frac{1}{2}a\left(\frac{5R_a^3 + 4\left(-1+\mu_i^3\right)R_i^3}{R_a^3\left(\frac{R_a^3+\left(-1+\mu_i^3\right)R_i^3}{R_a^3}\right)^{4/3}} - \frac{1}{\mu_i^4} - \frac{4}{\mu_i}\right) \tag{161}$$

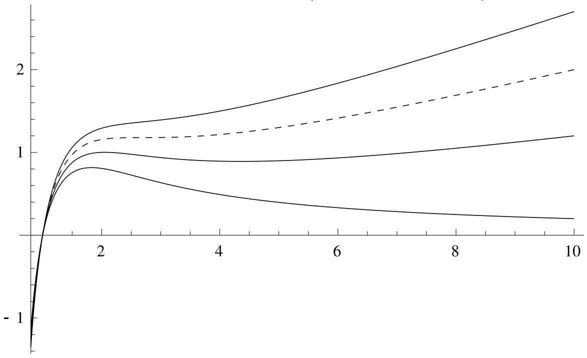
We can plot the latter function for different material parameters a and b. For a qualitative examination, it is sufficient to take $a = 1$ fixed, and vary the parameter b. Before we do so it is a good idea to compare the coefficients a and b for small strains with Hookes law. With $\boldsymbol{B} = \boldsymbol{F}\boldsymbol{F}^T$, $\boldsymbol{F} = \boldsymbol{H} + \boldsymbol{I}$, $\boldsymbol{E} = \frac{1}{2}(\boldsymbol{H} + \boldsymbol{H}^T)$, the linearised approximation of an inverse $(\boldsymbol{I} + \boldsymbol{A})^{-1} \approx \boldsymbol{I} - \boldsymbol{A}$ and Hookes law in terms of Lamés constants $\boldsymbol{T} = \lambda \mathrm{tr}(\boldsymbol{E})\boldsymbol{I} + 2G\boldsymbol{E}$ we find $G = a - b$. A shear modulus smaller or equal to zero is not reasonable, therefore we consider $b \leq a$. The next three plots give $p_i(\mu_i)$ for different values of $b = \{1, 0.5, 0, -0,5, -1\}$, $a = 1$ and $R_a/R_i = 2$. In the first graph one can see that for $b = 1 = a$ the function $p_i(\mu_i)$ has a zero slope at $\{\mu_i = 1, p_i = 0\}$, which corresponds to the shear modulus $G = 0$. With decreasing b, the initial slope becomes larger. In the second graph we can see that for $b > 0$ the p_i becomes negative for large μ_i, which is not reasonable.



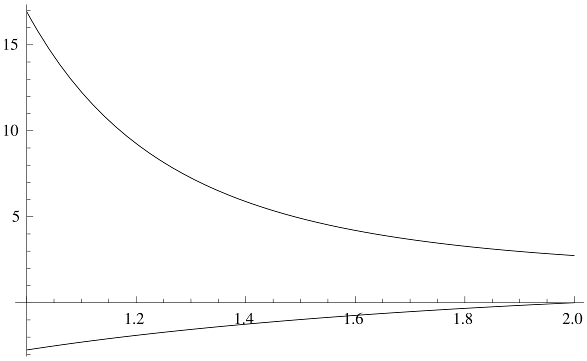


The following graph gives again p_i over μ_i. Here we can see that for $b = 0$ the pressure p_i approaches 0 as

$\mu_i \to \infty$. For the specific choice of $R_a/R_i = 2$, the parameter $b \approx -0.179904$ (dashed line, $b = -0.1$ and $b = -0.25$ for lower and upper curve, respectively) separates the nonmonotonic $p_i(\mu_i)$ curves ($b > -0.179904$) from the monotonic curves ($b < -0.179904$).



This results can be extracted by analysing the function (161), which is quite tedious. Next, let us have a look at the Cauchy stresses. The following graph gives the tangential and the radial stresses for the parameters $R_a/R_i = 2$, $R_i = 1$, $\mu_i = 2$, $a = 1$, $b = -1$. The radial stresses vanish at R_a due to the boundary condition $p_a = 0$.

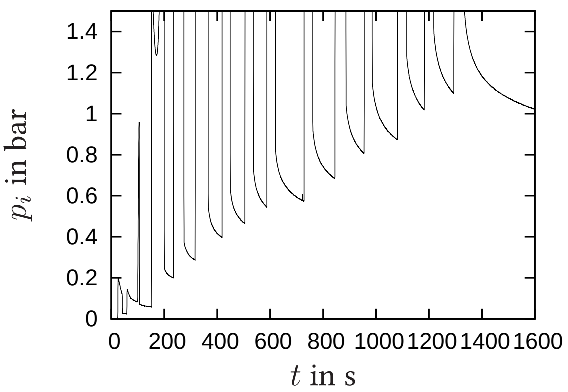


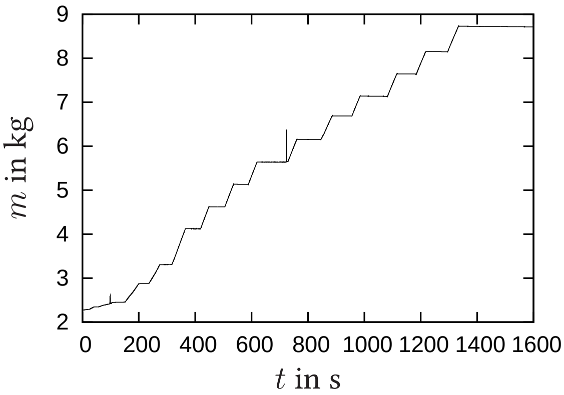
The graphs have been created using the following Mathematica-Skript:

```
Remove["Global‘*"];
Ra = 2; Ri = 1; a = 1;
mua = (1 + Ri^3/Ra^3*(mui^3 - 1))^(1/3);
pi[b_] = Simplify[-(2*a*(mui^-1 - mua^-1) + a*(mui^-4 - mua^-4)/2 +
      2*b*(mui - mua) - b*(mui^-2 - mua^-2))];
start = 0.8;
Plot[{pi[1], pi[0.5], pi[0], pi[-0.5], pi[-1]}, {mui, start, 1.2},
 PlotStyle -> {Black}, AxesOrigin -> {start, 0}]
start = 0.8;
Plot[{pi[1], pi[0.5], pi[0], pi[-0.5], pi[-1]}, {mui, start, 5},
 PlotStyle -> {Black}, AxesOrigin -> {start, 0}]
start = 0.8;
Plot[{pi[-0.179904], pi[0.0], pi[-0.1], pi[-0.25]}, {mui, start, 10},
 PlotStyle -> {{Black, Dashed}, {Black}, {Black}, {Black}},
 AxesOrigin -> {start, 0}]
Remove["Global‘*"];
mui = 2; Ri = 1; Ra = 2; a = 1; b = -1.0;
mu[R] = (1 + Ri^3/R^3*(mui^3 - 1))^(1/3);
trr[R_] = 2*a*mu[R]^-1 + a*mu[R]^-4/2 + 2*b*mu[R] - b*mu[R]^-2 - Const;
ttt[R_] =
  a*mu[R]^2 + b*mu[R]^-2 + 2*a*mu[R]^-1 - a*mu[R]^-4/2 - b*mu[R]^4 +
   2*b*mu[R] - b*mu[R]^-2 - Const;
```

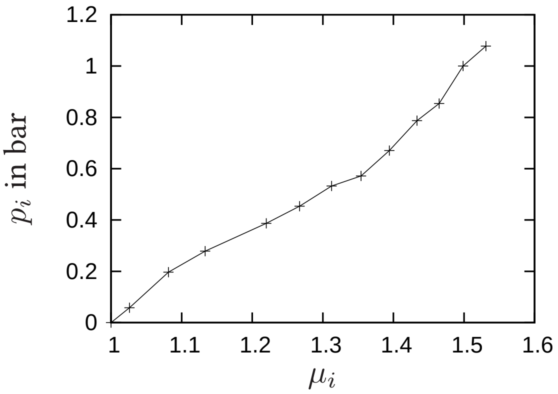
```
erg = Solve[{trr[Ra] == 0}, Const];
Const = Const /. erg[[1]];
Plot[{trr[R], ttt[R]}, {R, Ri, Ra}, PlotStyle -> {Black}]
```

It is more interesting to compare to experimental results. For this purpose, a rubber ball has been blown up using water. We have measured the mass of the ball and the pressure, and found a strong relaxation behaviour of the $p_i(t)$ curve. Therefore, at some instants, the mass has been held constant for a while, and the $p(t)$ curve has been recorded. Then, the relaxation behaviour has been approximated by a declining exponential function plus a constant p_∞ which is approached by $p(t)$ as $t \to \infty$. The parameters have been adopted to the measured curve by the least squares method, which allowed finally to assign a p_∞ to each mass, which is easily transformed into an r and μ. This is what we found:





On the left we see the measured $p(t)$ curve, on the right we see the measured mass of the ball $m(t)$. When the valve was open and water flowed in, the pressure increased up to 6 bar, therefore the graph has been chopped at 1.5 bar. The relaxation curves correspond to the times when the valve was closed. One can see that for the final measurement the waiting time was much longer than for the other points. This final point indicates a decreasing pressure, which may be due to damage, the theoretical findings of a purely elastic decreasing $p_i(\mu_i)$ or the long waiting time. Apparently we did not damage the ball. It is to suspect that the waiting time has a strong influence on the result, even if the relaxation behaviour is estimated. Thus, we left the last measurement point out on creating the following graph of $p_i(\mu_i)$. Taking the measurement error into account, we can speak of a linear $p_i(\mu_i)$ curve.

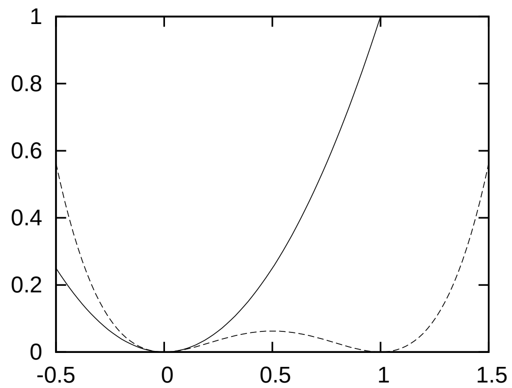


5 Nonconvex elastic energy

Task: Consider the elastostatic equilibrium of an elongated bar with the two strain energies

$$w_1 = \frac{E}{2}\varepsilon^2, \tag{162}$$
$$w_2 = \frac{E}{2}\varepsilon^2(\varepsilon - 1)^2, \tag{163}$$

illustrated in the following graph (w_i over ε with $E = 1$).



The displacement-strain relation is given as

$$\varepsilon = u', \tag{164}$$

where a prime indicates the derivative with respect to the coordinate x. The local equilibrium condition (static, no body forces) is

$$\sigma' = 0, \tag{165}$$

the stress-strain relation is given by

$$\sigma = \frac{\partial w}{\partial \varepsilon}. \tag{166}$$

The boundary conditions are $u(0) = 0$ and $u(l) = \Delta l$.

Solution: For w_1, the resulting differential equation and solution is

$$u'' = 0, \qquad u_1 = mx + n. \tag{167}$$

It is no problem to adapt m and n to the boundary conditions,

$$m = \frac{\Delta l}{l}, \qquad n = 0. \tag{168}$$

This is the non-surprising result for a linear elastic material. The situation is different for w_2, which results in the differential equation

$$0 = u''(u'^2 - u' + \frac{1}{6}). \tag{169}$$

Additional to the solution u_1, the latter DE is solved by

$$u'_{2\pm} = \frac{1}{2}(1 \pm \frac{1}{\sqrt{3}}), \tag{170}$$

which gives

$$u_+ = C_+ x + C_1, \qquad C_+ = \frac{1}{2}(1 + \frac{1}{\sqrt{3}}) \tag{171}$$

$$u_- = C_- x + C_2, \qquad C_- = \frac{1}{2}(1 - \frac{1}{\sqrt{3}}). \tag{172}$$

Interestingly, the strain is fixed to a specific value for this solutions. Clearly, none of the two additional solutions can be adopted alone to the boundary conditions, since only one constant may be adopted, but two boundary conditions need to be satisfied. But it is possible to glue the solutions $u_\pm$ piecewise together at some x_g, with the kinematic compatibility condition $u_+(x_\mathrm{g}) = u_-(x_\mathrm{g})$. *One* possibility is the approach

$$u = u_+ \text{ for } 0 < x < x_\mathrm{g} \tag{173}$$

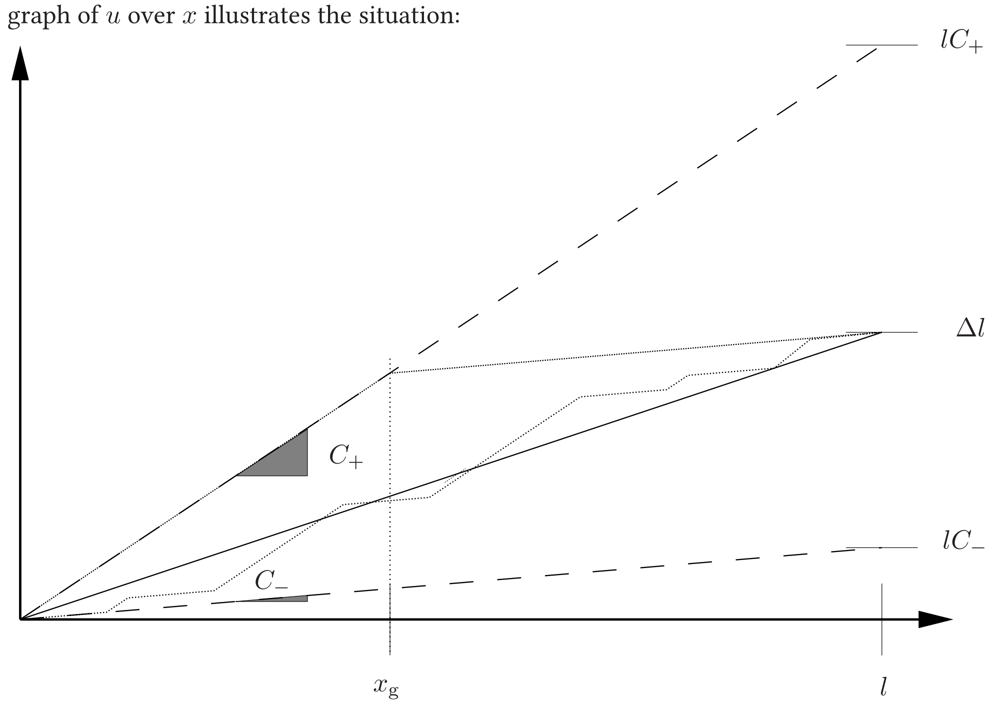
$$u = u_- \text{ for } x_\mathrm{g} < x < l, \tag{174}$$

where the constants x_g, C_1, C_2 are determined by the kinematic compatibility condition and the two boundary conditions,

$$C_1 = 0 \qquad C_2 = \Delta l - C_- l \qquad x_\mathrm{g} = \frac{\Delta l - C_- l}{C_+ - C_-}. \tag{175}$$

One may note that this solution can only be established if $0 < x_\mathrm{g} < l$. This inequality gives $lC_- < \Delta l < lC_+$. Thus, the additional solutions are only obtained for specific boundary condtions.

In this way, we are able to construct arbitrary many solutions to the elastostatic boundary value problem. The following graph of u over x illustrates the situation:

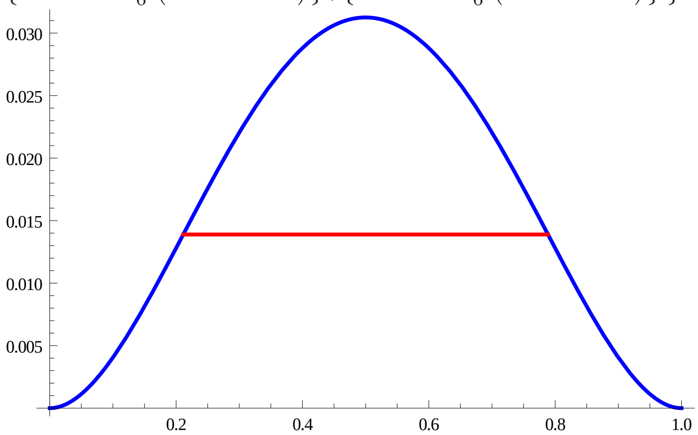


It is interesting to calculate the global strain energy:

```
Remove["Global`*"]
uplus = Cplus*x + C1;
uminus = Cminus*x + C2;
xg = (DeltaL - Cminus*L)/(Cplus - Cminus);
C1 = 0;
```

```
C2 = DeltaL - Cminus*L;
Cplus = 1/2*(1 + 1/Sqrt[3]);
Cminus = 1/2*(1 - 1/Sqrt[3]);
Print["Heterogene Gesamtenergie:"]
Whetero =
 A*FullSimplify[
   Integrate[EMO/2*Cplus^2*(1 - Cplus)^2, {x, 0, xg}] +
    Integrate[EMO/2*Cminus^2*(1 - Cminus)^2, {x, xg, L}]]
Print["Homogene Gesamtenergie:"]
Whomo = A*
  FullSimplify[
   Integrate[EMO/2*(DeltaL/L)^2*(1 - DeltaL/L)^2, {x, 0, L}]]
Print["Bei welchem DeltaL haben beide Loesungen gleich viel Energie?"]
erg = Solve[Whetero - Whomo == 0, DeltaL]
L = 1; A = 1; EMO = 1;
p1 = Plot[Whetero, {DeltaL, erg[[1, 1, 2]], erg[[2, 1, 2]]},
   PlotStyle -> {Red, Thick}];
p2 = Plot[Whomo, {DeltaL, 0, 1}, PlotStyle -> {Blue, Thick}];
Show[p2, p1]
```

Heterogene Gesamtenergie:
$\frac{A\mathrm{EMO}L}{72}$
Homogene Gesamtenergie:
$\frac{A\mathrm{DeltaL}^2\mathrm{EMO}(\mathrm{DeltaL}-L)^2}{2L^3}$
Bei welchem DeltaL haben beide Loesungen gleich viel Energie?
$\left\{\left\{\mathrm{DeltaL}\to\frac{1}{6}\left(3L-\sqrt{3}L\right)\right\},\left\{\mathrm{DeltaL}\to\frac{1}{6}\left(3L+\sqrt{3}L\right)\right\},\right.$
$\left.\left\{\mathrm{DeltaL}\to\frac{1}{6}\left(3L-\sqrt{15}L\right)\right\},\left\{\mathrm{DeltaL}\to\frac{1}{6}\left(3L+\sqrt{15}L\right)\right\}\right\}$



There are 4 solutions, which are the intersections of a W-shaped function with a horizontal line. Interestingly, the heterogeneous global strain energy is independent of Δl. The heterogeneous solution becomes preferrable at

$$\Delta l = \frac{l}{2}(1-\frac{1}{\sqrt{3}}) = lC_-, \quad \Delta l = \frac{l}{2}(1+\frac{1}{\sqrt{3}}) = lC_+, \quad \Delta l = \frac{l}{2}(1-\frac{\sqrt{5}}{\sqrt{3}}), \quad \Delta l = \frac{l}{2}(1+\frac{\sqrt{5}}{\sqrt{3}}). \tag{176}$$

In the interval $lC_- < \Delta l < lC_+$, the heterogeneous solution has the lower global strain energy, which means that just in the range where we can easily construct heterogeneous solutions, these are energetically favourable. Investigating the situation, one finds that the additional solutions are connected to the points where $\partial^2 w/\partial\varepsilon^2 = 0$. Thus, we can exclude such behaviour by ensuring that $\partial^2 w/\partial\varepsilon^2 > 0\ \forall\varepsilon$, i.e. that w is convex in ε. The case that w is concave in ε must be excluded, since this would imply that w is negative for some ε. The latter

statement is only valid if the strain measure ε is linear in u: for nonlinear strain measures, the situation is more complicated.
A crucial point is the kind of function that we expect as a solution for the displacement field. We have assumed that the displacements are C_0-continuous. One might as well require u to be a smooth function, excluding the piecewise linear solution. Looking at real material behavior, it appears that u may indeed be a piecewise continuous function. Only if we look very deep at the transition point into the material, we may argue that the displacement field is smooth. However, one is then close to the atomic scale, where continuum mechanics is not meant to be applied.
Unfortunately, one cannot easily generalize this findings to the multidimensional case. Consider, for example, the C_0 continuity of the displacement field: The first derivative of $\boldsymbol{u}(\boldsymbol{x})$ may have a jump, but we have to specify in which direction, referred to as $\boldsymbol{n}$. In 1D, there is only one direction. This is done by the directional derivative

$$\boldsymbol{u}'(\boldsymbol{n}) = \lim_{\delta\to 0} \frac{1}{\delta} \boldsymbol{u}(\boldsymbol{x}_0 + \delta \boldsymbol{n}) - \boldsymbol{u}(\boldsymbol{x}_0), \tag{177}$$

which can be written as

$$\boldsymbol{u}'(\boldsymbol{n}) = \boldsymbol{u} \otimes \nabla_0 \cdot \boldsymbol{n}. \tag{178}$$

If the derivative of the displacement field undergoes a jump in direction of $\boldsymbol{n}$, we have a singular surface with the normal $\boldsymbol{n}$. If inside the surface, i.e. in directions perpendicular to $\boldsymbol{n}$ there is no jump, we can conclude the jump balance of the displacement gradient to look like

$$\boldsymbol{H}^+ - \boldsymbol{H}^- = \boldsymbol{a} \otimes \boldsymbol{n} = \boldsymbol{F}^+ - \boldsymbol{F}^-, \tag{179}$$

where $\boldsymbol{a} = \boldsymbol{u}'(\boldsymbol{n})$. If we have two jumps, say in direction $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$, we speak of an singular edge, where two singular surfaces meet. Three jump directions are not possible, presuming that we don't have singular material points that are detached from the body.
The jump of $\boldsymbol{F}$ being a rank-one tensor at a singular surface induces the so called rank-one-convexity. It appears that a strain energy that satisfies

$$\forall\ \boldsymbol{a}, \boldsymbol{b} \in R^3 \backslash \{\boldsymbol{0}\} :\ 0 < (\boldsymbol{a} \otimes \boldsymbol{b}) \cdot \cdot w_{,\boldsymbol{FF}} \cdot \cdot (\boldsymbol{a} \otimes \boldsymbol{b}) \tag{180}$$

excludes heterogeneous solutions. There is a tremendous amount peculiarities in this topic, we shall stop here. The following tasks are related to this issues:
Apply the notion of strict convexity to a strain energy $w(\boldsymbol{F})$ and examine (1) the rotation of a body and (2) an isochoric plane strain deformation.

Collect all constraints that should be imposed on an elastic strain energy. Consider mathematical and physical necessities.

Assume that w is a function of $\boldsymbol{C}$, and expand the derivative $w_{,\boldsymbol{FF}}$ by the chain rule. Determine the derivatives $\boldsymbol{C}_{,\boldsymbol{F}}$ and $\boldsymbol{C}_{,\boldsymbol{FF}}$.

6 Convexity

For the problem *finding the minimum value of a real function* to have a unique solution, convexity of the function is sufficient but not necessary. What is necessary and sufficient is monotonicity.
Likewise, to have a unique solution to the variational problem of minimizing the global elastic energy, convexity in $\boldsymbol{F}$ of the strain energy is asked to much. **Task:** Show that a $w(\boldsymbol{F})$ can not be objective if w is convex in $\boldsymbol{F}$.
Solution: We require $w(\boldsymbol{Q}) = 0$ for all $\boldsymbol{Q} \in Orth^+$. Convexity in $\boldsymbol{F}$ requires

$$w(\alpha \boldsymbol{F}_1 + (1-\alpha)\boldsymbol{F}_2) < w(\alpha \boldsymbol{F}_1) + w((1-\alpha)\boldsymbol{F}_2) \tag{181}$$

for all $\boldsymbol{F}_i$ and $0 < \alpha < 1$. Plugging in two arbitrary orthogonal tensors in $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ shows the contradiction.
What one actually needs from w is that the BVP has a unique solution for all possible domains and BC. This is termed *quasiconvexity*. It is hard to verify analytically. However, it is somewhere between polyconvexity and rank-1-convexity, which both can be checked more easily. **Task:** Show that $w = E\|\boldsymbol{C}\|^2/2$ is r1c.
Solution: Take the first and second derivative:

$$w,_{\boldsymbol{F}} = 2E\boldsymbol{F}\boldsymbol{F}^T\boldsymbol{F} = \boldsymbol{T}^{1PK} \tag{182}$$

$$w,_{\boldsymbol{F}\boldsymbol{F}} = 2E\left[\delta_{np}F_{oj}F_{mj} + F_{on}F_{mp} + F_{kn}F_{kp}\delta mo\right] \boldsymbol{e}_m \otimes \boldsymbol{e}_n \otimes \boldsymbol{e}_o \otimes \boldsymbol{e}_p. \tag{183}$$

Scalar product with $\boldsymbol{b}$ ($\|\boldsymbol{b}\| \neq \boldsymbol{0}$, $\boldsymbol{b}^* = \boldsymbol{F}\boldsymbol{b}$) with the second and fourth index gives the acoustic tensor $\boldsymbol{A}$,

$$\boldsymbol{A} = 2E[\boldsymbol{F}\boldsymbol{F}^T + \boldsymbol{b}^* \otimes \boldsymbol{b}^* + (\boldsymbol{b}^* \cdot \boldsymbol{b}^*)\boldsymbol{I}]. \tag{184}$$

The summands are all positive definit or positive semidefinit, hence $\boldsymbol{A}$ is positive definit. Thus, the remaining contractions with $\boldsymbol{a}$ are always greater than zero. In conclusion, stress-strain curves for monotonic tests should always be monotonic when this strain energy is used.
Tasks: Plot the stress-strain-curve for some characteristic tests. Check w for polyconvexity. **Solution:**

$$\boldsymbol{T} = J^{-1}\boldsymbol{B}^2 \tag{185}$$

.

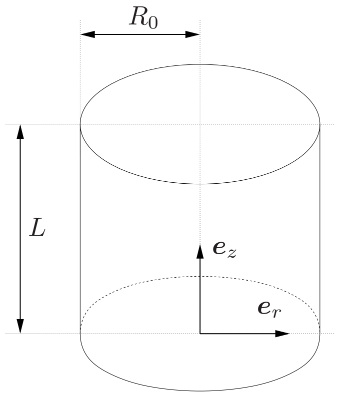
7 The Poynting effect

Consider a cylindical specimen of radius R_0 and length L subjected to torsion, i.e. the base planes are rotated relative to each other, but their distance is kept fixed, i.e. no axial elongation or contraction is allowed. The outer surface is free of traction. The material is assumed to be hyperelastic, incompressible and isotropic, with the elastic energy $w(\mathrm{I}, \mathbb{I})$ only a function of the first and second invariant of $\boldsymbol{B} = \boldsymbol{F}\boldsymbol{F}^T$. From representation theory (Ariano and Rivlin, theorem 7.12 in Bertrams book), the stress strain relation is known to be

$$\boldsymbol{T} = -p\boldsymbol{I} + 2(w_{\mathrm{I}}\boldsymbol{B} - w_{\mathbb{I}}\boldsymbol{B}^{-1}), \tag{186}$$

where the first summand is the reaction part of the stresses due to the incompressibility condition, and the second summand the extra part of the stresses, which depends on the deformation. The indices I and $\mathbb{I}$ at w indicate the partial derivatives w.r.t. the invariants I and $\mathbb{I}$. We have put for convenience the constant mass density ρ_0 into the strain energy w, i.e., w is volume-specific.

Task: Determine the axial force that needs to be applied on the base planes such that the length is kept constant and the torsion moment as a function of the angle by which the base planes are rotated wrt one another.



Solution: Since the material is isochoric, and the length is kept constant, and the material is homogeneous, the deformation must be homogeneous, too. Thus, there is no lateral straining. In fact, this deformation belongs to family 3 of the universal solutions of incompressible solids (Section 8.2 in Bertrams book). Employing cylindrical coordinates, we approach the motion of the cylinder by

$$r = R, \tag{187}$$
$$\phi = \Phi + DZ, \tag{188}$$
$$z = Z, \tag{189}$$

with a constant torsion D, where the origion of the coordinate system lies in the lower base plane. The overall rotation angle between the base planes is $\theta = DL$. The nabla operator in cylindrical coordinates is

$$\nabla = \frac{\partial}{\partial r}\boldsymbol{e}_r + \frac{1}{r}\frac{\partial}{\partial \phi}\boldsymbol{e}_\phi + \frac{\partial}{\partial z}\boldsymbol{e}_z, \tag{190}$$

the base vectors $\boldsymbol{e}_r$ and $\boldsymbol{e}_\phi$ depend on ϕ,

$$\boldsymbol{e}_{r,\phi} = \boldsymbol{e}_\phi \tag{191}$$
$$\boldsymbol{e}_{\phi,\phi} = -\boldsymbol{e}_r. \tag{192}$$

Remembering that the nabla operator must be taken w.r.t. the material coordinates R, Φ, Z in order to determine $\boldsymbol{F} = \boldsymbol{x} \otimes \nabla$ gives, with $\boldsymbol{x} = r(R, \Phi, Z)\boldsymbol{e}_r(\phi(R, \Phi, Z)) + z(R, \Phi, Z)\boldsymbol{e}_z$

$$\boldsymbol{F} = \boldsymbol{e}_r \otimes \boldsymbol{e}_R + \boldsymbol{e}_\phi \otimes \boldsymbol{e}_\Phi + \boldsymbol{e}_z \otimes \boldsymbol{e}_Z + DR\boldsymbol{e}_\phi \otimes \boldsymbol{e}_Z. \tag{193}$$

$\boldsymbol{B} = \boldsymbol{F}\boldsymbol{F}^T$ and $\boldsymbol{B}^{-1}$ are determined easily,

$$\boldsymbol{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 + D^2R^2 & DR \\ 0 & DR & 1 \end{bmatrix} \boldsymbol{e}_i \otimes \boldsymbol{e}_j, \qquad \boldsymbol{B}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -DR \\ 0 & -DR & 1 + D^2R^2 \end{bmatrix} \boldsymbol{e}_i \otimes \boldsymbol{e}_j. \tag{194}$$

Note that the invariants of $\boldsymbol{B}$ are

$$\mathrm{I} = \mathbb{I} = 3 + D^2R^2, \tag{195}$$

which induces the equalities

$$\frac{\mathrm{dI}}{\mathrm{d}R} = \frac{\mathrm{d}\mathbb{I}}{\mathrm{d}R} = 2D^2R \tag{196}$$

$$\mathrm{d}R = \frac{1}{2D^2R}\mathrm{dI} = \frac{1}{2D^2R}\mathrm{d}\mathbb{I}. \tag{197}$$

We will make use of this peculiarity of the deformation later on. The stresses result as

$$\boldsymbol{T} = -p(R)\boldsymbol{I} + 2\begin{bmatrix} w_{\mathrm{I}} - w_{\mathbb{I}} & 0 & 0 \\ +0 & w_{\mathrm{I}}(1 + D^2R^2) - w_{\mathbb{I}} & (w_{\mathrm{I}} + w_{\mathbb{I}})DR \\ 0 & (w_{\mathrm{I}} + w_{\mathbb{I}})DR & w_{\mathrm{I}} - w_{\mathbb{I}}(1 + D^2R^2) \end{bmatrix} \boldsymbol{e}_i \otimes \boldsymbol{e}_j. \tag{198}$$

The divergence $\boldsymbol{T} \cdot \nabla$ must vanish. The evaluation is not difficult, since many scalar products and derivatives are zero. It remains only

$$0 = \frac{\partial T_{rr}}{\partial r} + \frac{T_{rr} - T_{\phi\phi}}{r} \tag{199}$$

in direction of $\boldsymbol{e}_r$. With with $r = R$, the symmetry of second derivatives $w_{\mathrm{I}\mathbb{I}} - w_{\mathbb{I}\mathrm{I}}$ and $\mathrm{dI}/\mathrm{d}R = \mathrm{d}\mathbb{I}/\mathrm{d}R$ we get

$$0 = -p'(R) + 4D^2R[w_{\mathrm{II}} - w_{\mathbb{I}\mathbb{I}}] - 2D^2Rw_{\mathrm{I}}. \tag{200}$$

The prime at p indicates the derivative w.r.t. to R. We can solve for $p(R)$ by integration,

$$p(R) = 2(w_{\mathrm{I}} - w_{\mathbb{I}}) - \int 2D^2Rw_{\mathrm{I}}\mathrm{d}R + C. \tag{201}$$

Unfortunately, only one of the integrals w.r.t. R, namely $\int 4D^2R(w_{\mathrm{II}} - w_{\mathbb{I}\mathbb{I}})\mathrm{d}R = 2(w_{\mathrm{I}} - w_{\mathbb{I}})$ can be solved for. It is tempting to replace the remaining $\mathrm{d}R$ by dI. This is, however, not sufficient. The complete differential w.r.t. R needs to be carried out like

$$\frac{\mathrm{d}w}{\mathrm{d}R} = \frac{\mathrm{d}w}{\mathrm{dI}}\frac{\mathrm{d}I}{\mathrm{d}R} + \frac{\mathrm{d}w}{\mathrm{d}\mathbb{I}}\frac{\mathrm{d}\mathbb{I}}{\mathrm{d}R} = w_{\mathrm{I}}\frac{\mathrm{d}I}{\mathrm{d}R} + w_{\mathbb{I}}\frac{\mathrm{d}II}{\mathrm{d}R}. \tag{202}$$

Replacing w_{I} just brings up $w_{\mathbb{I}}$, and vice versa.
One might want to replace the function $w(\mathrm{I}, \mathbb{I})$ by something like $w^*(\mathrm{I})$, since we have $\mathrm{I} = \mathbb{I}$ for this deformation, i.e. there is effectively only one argument for w, and not two. This is possible as long as one does not use the gradient, i.e. the expressions w_{I} and $w_{\mathbb{I}}$. These can not be distinguished by looking only at w^*.
In order to proceed, we need to make an assumption on w that allows to summarize further. Let us assume that we can write w like

$$w = w_1(\mathrm{I}) + w_2(\mathbb{I}). \tag{203}$$

Then w_{I} does not involve $\mathbb{I}$, which allows to replace

$$\int 2D^2Rw_{\mathrm{I}}\mathrm{d}R = w_1. \tag{204}$$

With this assumption, we can replace $w_{\mathrm{I}} = w_1'$ and $w_{\mathbb{I}} = w_2'$, where the prime indicates the derivative w.r.t. the argument. We obtain

$$p(R) = 2(w_{\mathrm{I}} - w_{\mathbb{I}}) - w_1 + C. \tag{205}$$

The boundary condition is $\boldsymbol{T}\boldsymbol{e}_r = \boldsymbol{0}$, which leaves only

$$\begin{aligned} 0 &= T_{rr}|_{R_0} && (206)\\ &= -p(R_0) + 2(w_{\mathrm{I}}|_{R_0} - w_{\mathbb{I}}|_{R_0}) && (207)\\ &= w_1|_{R_0} - C, && (208)\\ C &= w_1|_{R_0}. && (209)\end{aligned}$$

By this, $\boldsymbol{T}$ is determined completely. We can now calculate the axial force and torsion moment. First, the stress vector at the upper base plane is determined,

$$\boldsymbol{t} = \boldsymbol{T}\boldsymbol{e}_z = T_{\phi z}\boldsymbol{e}_\phi + T_{zz}\boldsymbol{e}_z. \tag{210}$$

There is one component parallel to $\boldsymbol{e}_z$ that is responsible for the axial force, and in direction of $\boldsymbol{e}_\phi$ that induces the torsion moment.

7.1 Axial force

Integrating z-component over the base plane gives the axial overall force,

$$F = \int_0^{R_0}\int_0^{2\pi}(w_1 - w_1|_{R_0} - 2D^2R^2w_2')R\mathrm{d}\phi\mathrm{d}R. \tag{211}$$

We can choose whether we replace $\mathrm{d}R$ by $\mathrm{dI}/(2D^2R)$ or $\mathrm{d}\mathbb{I}/(2D^2R)$, thanks to our assumption $w = w_1 + w_2$. We denote this by $\mathrm{d}(\mathrm{I}, \mathbb{I})$.

$$F = \frac{1}{2D^2}\int_3^{3+D^2R_0^2}\int_0^{2\pi}(w_1 - w_1|_{R_0} - 2D^2R^2w_2')\mathrm{d}\phi\mathrm{d}(\mathrm{I}, \mathbb{I}). \tag{212}$$

Further, we can replace R^2 by $(\mathbb{I} - 3)/D^2$, which gives

$$F = \frac{\pi}{D^2}\int_3^{3+D^2R_0^2}(w_1 - w_1|_{R_0} - 2(\mathbb{I} - 3)w_2')\mathrm{d}(\mathrm{I}, \mathbb{I}). \tag{213}$$

The term with $\mathbb{I}$ can be integrated by parts for $\mathbb{I}$,

$$\begin{aligned} F &= \frac{\pi}{D^2}\left[\int_3^{3+D^2R_0^2}(w_1 - w_1|_{R_0})\mathrm{dI} - \int_3^{3+D^2R_0^2}2w_2'(\mathbb{I} - 3))\mathrm{d}\mathbb{I}\right] && (214)\\ &= \frac{\pi}{D^2}\left[\int_3^{3+D^2R_0^2}(w_1 - w_1|_{R_0})\mathrm{dI} - [2w_2(\mathbb{I} - 3)]_3^{3+D^2R_0^2} + \int_3^{3+D^2R_0^2}2w_2\mathrm{d}\mathbb{I}\right] && (215)\\ &= \frac{\pi}{D^2}\left[\int_3^{3+D^2R_0^2}(w_1 - w_1|_{R_0})\mathrm{dI} - 2w_2|_{R_0}D^2R_0^2 + \int_3^{3+D^2R_0^2}2w_2\mathrm{d}\mathbb{I}\right] && (216)\\ &= \frac{\pi}{D^2}\left[\int_3^{3+D^2R_0^2}(w_1 - w_1|_{R_0})\mathrm{dI} + \int_3^{3+D^2R_0^2}2(w_2 - w_2|_{R_0})\mathrm{d}\mathbb{I}\right] && (217)\end{aligned}$$

Locally, the deformation is a simple shear. Let us make the reasonable assumption that the strain energy $w = w_1 + w_2$ increases monotonocally with the shear deformation. It is very likely that $w_1 + 2w_2$ then also increases monotonically with the shear deformation. Sufficient therefore is that w_1 and w_2 increase monotonically with I and $\mathbb{I}$, respectively. Truesdell firstly formulated this requirement to w_1 and w_2,

$$w_1' > 0, \qquad w_2' > 0. \tag{218}$$

Such equations are called *empirical inequalities.* They have been formulated by engineers before mathematicians imposed convexity requirements on w. Some of these inequalities, e.g. the Baker-Ericksen inequalities, have been derived later as necessary conditions for rank-1-convexity.
We see that the shear deformation increases monotonically with the radius R. We further notice that the radius increases monotonically with the invariant I or $\mathbb{I}$. Consequently, w must increase monotonically with R, i.e. it takes its maximum value at the maximum radius R_0. Therefore, the last integrals *must be negative.* The elongation in case of a free axial straining is called Poynting-effect. It is very common in rubber-like materials. Only for very exotic materials, a weak negative Poynting effect is reported [Mihai, L. A. and Goriely, A. (2011) Proceedings of the Royal Society]. It is an effect of second order (and thus hard to measure), which is demonstrated next.
We assume that w is normalized, i.e. at $R = 0$ we have $w = 0$. The most basic assumption that one can make is that w is quadratic in the shear, i.e. quadratic in R, i.e. quasi-linear in the invariants I and $\mathbb{I}$. This material law is the the Mooney-Rivlin energy, which is $w = \alpha(\mathrm{I} - 3)/2 + \beta(\mathbb{I} - 3)/2$. Evaluating the integrals gives

$$F = \frac{\pi}{D^2}\left[\int_3^{3+D^2R_0^2} \frac{\alpha}{2}[\mathrm{I} - 3 - D^2R_0^2]\mathrm{dI} + \int_3^{3+D^2R_0^2} \beta[\mathbb{I} - 3 - D^2R_0^2]\mathrm{d}\mathbb{I}\right] \tag{219}$$

$$= -\left(\frac{\alpha}{2} + \beta\right)\frac{\pi D^2 R_0^4}{2}. \tag{220}$$

We now see that the effect is of second order in D in this very basic case, i.e. it appears only at large deformations. For the Mooney-Rivlin energy, only the sum of $\alpha/2$ and β enters F. Suppose we want to determine α *and* β. We need another measurement, namely the torsion moment, which we consider next.

7.2 Torsion moment

We now determine the torsion moment that needs to be applied. This is achieved by multiplying the $\boldsymbol{e}_\phi$-component of $\boldsymbol{t} = \boldsymbol{T}\boldsymbol{e}_z$ by R, and integration over the base plane, i.e. we find

$$M = 2D\int_0^{R_0}\int_0^{2\pi}(w_{\mathrm{I}} + w_{\mathbb{I}})R^3\mathrm{d}\phi\mathrm{d}R \tag{221}$$

With $w_{\mathrm{I}} + w_{\mathbb{I}} = \frac{\mathrm{d}w}{\mathrm{d}R}2D^2R$ we can rewrite this,

$$M = \frac{2\pi}{D}\int_0^{R_0}\frac{\mathrm{d}w}{\mathrm{d}R}R^2\mathrm{d}R. \tag{222}$$

We can already see that M is greater than zero for w growing monotonic with R, independently on the decomposition assumption that we needed for the determination of the axial force. This is due to the fact that the torsion moment does not involve $p(R)$. Remember that we needed the assumption $w(\mathrm{I}, \mathbb{I}) = w_1(\mathrm{I}) + w_2(\mathbb{I})$ to integrate $p'(R)$ for R.
Evaluating the latter equation for the Mooney-Rivlin material gives

$$M = \pi(\alpha + \beta)DR_0^4/2. \tag{223}$$

Comparing with the linear beam theory, we can identify the known result

$$M = GDI_p, \quad I_p = \pi R_0^4/2, \tag{224}$$

where I_p is the polar second moment of area and $G = \alpha + \beta$ the relation between the shear modulus and α and β for the Mooney-Rivlin-material for small strains.

7.3 Discussion

When one measures the curves $F(D)$ and $M(D)$, one measures only along $\mathrm{I} = \mathrm{II}$ in both cases. Independently, the results $F(D)$ and $M(D)$ allow for identifying w. While measuring the axial force allows to access w_{I}, the measurement of M gives access to $\mathrm{d}w/\mathrm{d}R$, i.e. the sum of w_{I} and w_{II}. Thus, one can determine not only w, but also the gradient w_{I} and w_{II}. All together, the torsion test is a good method to identify strain energies of isotropic, incompressible materials.

8 The Prandtl-Reuss-equations

An elastoplastic material law is given by

$$\dot{\boldsymbol{T}}' = 2G[\dot{\boldsymbol{E}}' - \frac{3}{2\sigma_f^2}(\boldsymbol{T}' \cdot\cdot \dot{\boldsymbol{E}})\boldsymbol{T}'] \qquad \text{if } \phi = 0 \text{ and } B > 0,\ else \tag{225}$$

$$\dot{\boldsymbol{T}}' = 2G\dot{\boldsymbol{E}}' \tag{226}$$

$$\mathrm{tr}\boldsymbol{T} = 3K\mathrm{tr}\boldsymbol{E} \tag{227}$$

$$\phi = \boldsymbol{T}' \cdot\cdot \boldsymbol{T}' - \frac{2}{3}\sigma_f \tag{228}$$

$$B = \boldsymbol{T}' \cdot\cdot \dot{\boldsymbol{E}}. \tag{229}$$

The latter set of equation characterizes an ideal plastic, isotropic von Mises material. It is time independent in the sense that the result of a process does not depend on the speed at which it is carried out. The time serves merely as a process parameter. G and K are the two independent elasticity constants, σ_f is the flow stress, ϕ is the flow criterion, and B is the loading condition. B is obtained by considering $\dot{\phi}$ for a purely elastic process, i.e. one has

$$\dot{\phi} = \frac{\partial\phi}{\partial\boldsymbol{T}} \cdot\cdot \frac{\partial\boldsymbol{T}}{\partial t} \tag{230}$$

$$= 2\boldsymbol{T}' \cdot\cdot \dot{\boldsymbol{T}}. \tag{231}$$

Recognizing that only the deviatoric part of $\dot{\boldsymbol{T}}$ matters, and inserting the flow rule in case of elasticity gives

$$\dot{\phi} = 4G\boldsymbol{T}' \cdot\cdot \dot{\boldsymbol{E}}, \tag{232}$$

where the $4G$ can be blanked out since we are only interested in the sign. The following task is not specifically a nonlinear continuum mechanics task, since no displacement-strain-relation $\boldsymbol{E}(\boldsymbol{u})$ needed.
Task: Process 1: Starting from the strain- and stress free material, consider a uniaxial tension test for $0 < t < 1\mathrm{s}$, where $\boldsymbol{T}(t = 1\mathrm{s}) = \sigma_f \boldsymbol{e}_1 \otimes \boldsymbol{e}_1$, with Youngs modulus E. Then, in $1\mathrm{s} < t < 2\mathrm{s}$ a shear test is imposed, with $\dot{\boldsymbol{E}} = \frac{1}{\mathrm{s}}\frac{\sigma_f}{2\sqrt{3}G}(\boldsymbol{e}_1 \otimes \boldsymbol{e}_2 + \boldsymbol{e}_2 \otimes \boldsymbol{e}_1)$. Give $\boldsymbol{T}(t)$ and $\boldsymbol{E}(t)$ for this process. Process 2: Then consider a process where firstly the shear test from the last process is imposed, followed by a tension test without lateral straining, i.e. $\dot{\boldsymbol{E}} = \frac{\sigma_f}{E}\boldsymbol{e}_1 \otimes \boldsymbol{e}_1$, and determine again $\boldsymbol{T}(t)$ and $\boldsymbol{E}(t)$. Both processes are carried out with a constant strain rate.
Solution: Process 1: One can see easily that $\phi = 0$ at $\boldsymbol{T} = \sigma_f \boldsymbol{e}_1 \otimes \boldsymbol{e}_1$, which means that the first step of process 1 brings the material to the yield limit. We check whether the next step is elastic or plastic by evaluating B,

$$B = \sigma_f \boldsymbol{e}_1 \otimes \boldsymbol{e}_1 \cdot\cdot \frac{1}{\mathrm{s}}\frac{\sigma_f}{2\sqrt{3}G}(\boldsymbol{e}_1 \otimes \boldsymbol{e}_2 + \boldsymbol{e}_2 \otimes \boldsymbol{e}_1) = 0. \tag{233}$$

The next step is initially neutral with respect to the loading. However, one can see that, if we impose a small elastic load increment by $\dot{\boldsymbol{T}}' = 2G\dot{\boldsymbol{E}}'$, $B > 0$ for the next load increment. When $\dot{\phi} = 0$ one can also take a look $\ddot{\phi}$ to determine whether the material is going to be deformed plastically,

$$\ddot{\phi} = 4G(\dot{\boldsymbol{T}}' \cdot\cdot \dot{\boldsymbol{E}} + \boldsymbol{T}' \cdot\cdot \ddot{\boldsymbol{E}}). \tag{234}$$

The quantity $\ddot{\boldsymbol{E}}$ is undefined when the strain path changes, and $\boldsymbol{0}$ during the loading, since we presume a constant strain rate. Consequently, at the beginning of the second step we have $\ddot{\phi} = 8G^2\dot{\boldsymbol{E}}' \cdot\cdot \dot{\boldsymbol{E}} > 0$. Therefore, we have to integrate the stresses following eq. (225),

$$\dot{\boldsymbol{T}}' = 2G[\frac{1}{\mathrm{s}}\frac{\sigma_f}{2\sqrt{3}G}(\boldsymbol{e}_1 \otimes \boldsymbol{e}_2 + \boldsymbol{e}_2 \otimes \boldsymbol{e}_1) - \frac{3}{2\sigma_f^2}(\boldsymbol{T}' \cdot\cdot \frac{1}{\mathrm{s}}\frac{\sigma_f}{2\sqrt{3}G}(\boldsymbol{e}_1 \otimes \boldsymbol{e}_2 + \boldsymbol{e}_2 \otimes \boldsymbol{e}_1))\boldsymbol{T}'] \tag{235}$$

$$= \frac{1}{\mathrm{s}}\frac{\sigma_f}{\sqrt{3}}(\boldsymbol{e}_1 \otimes \boldsymbol{e}_2 + \boldsymbol{e}_2 \otimes \boldsymbol{e}_1) - \frac{3}{2\sigma_f^2}(\frac{1}{\mathrm{s}}\frac{\sigma_f}{\sqrt{3}}2T_{12})\boldsymbol{T}' \tag{236}$$

$$= \frac{1}{\mathrm{s}}\frac{\sigma_f}{\sqrt{3}}(\boldsymbol{e}_1 \otimes \boldsymbol{e}_2 + \boldsymbol{e}_2 \otimes \boldsymbol{e}_1) - \frac{1}{\mathrm{s}}\frac{\sqrt{3}}{\sigma_f}T_{12}\boldsymbol{T}'. \tag{237}$$

This gives a system of six differential equations for the components of $\boldsymbol{T}'$,

$$\dot{T}'_{12} = \frac{1}{s}\left(\frac{\sigma_f}{\sqrt{3}} - \frac{\sqrt{3}}{\sigma_f}T'^2_{12}\right) \tag{238}$$

$$\dot{T}'_{ij} = -\frac{1}{s}\frac{\sqrt{3}}{\sigma_f}T'_{12}T'_{ij} \qquad \text{for } (i,j) \neq \{(1,2),(2,1)\}, \tag{239}$$

with the initial condition $\boldsymbol{T}(t = 1\text{s}) = \sigma_f \boldsymbol{e}_1 \otimes \boldsymbol{e}_1$. In order to deal with the ODE it is convenient to introduce the step time $t_2 = t - 1\text{s}$, which may be substituted backwards later on. Solving firstly for $T_{12}(t_2)$ gives

$$T'_{12}(t_2) = \frac{\sigma_f}{\sqrt{3}}\tanh(t_2 + c), \tag{240}$$

where $c = 0$ is found due to $T_{12}(t_2 = 0) = 0$. This enables us to give solutions for the remaining components, the ODE of which are now

$$\dot{T}'_{ij} = -\frac{1}{s}\tanh(t_2)T'_{ij} \qquad \text{for } (i,j) \neq \{(1,2),(2,1)\}, \tag{241}$$

with the solutions

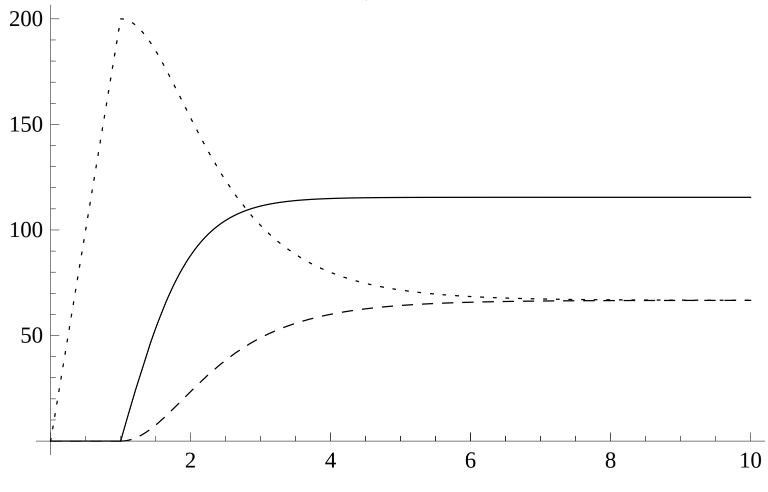
$$T'_{ij} = \frac{c_{ij}}{\cosh(t_2)} \qquad \text{for } (i,j) \neq \{(1,2),(2,1)\}, \tag{242}$$

With $\cosh(0) = 1$ it is clear that $c_{ij} = T'_{ij}(t_2 = 0)$, which means that all components which start of at zero remain zero, which holds for $T_{13} = T_{31}$ and $T_{23} = T_{32}$. The remaining components give $c_{11} = 2\sigma_f/3$ and $c_{22} = c_{33} = -\sigma_f/3$. Since the shear deformation is deviatoric, the volumetric parts of $\boldsymbol{E}$ and $\boldsymbol{T}$ remain constant during the second step, and we can assemble the stress tensor in the second step by adding $\sigma_f/3\boldsymbol{I}$ to the latter findings. Finally, we have

$$\boldsymbol{T}(t_2) = \frac{\sigma_f}{3}\Bigg[(1 + \frac{2}{\cosh(t_2)})\boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + (1 - \frac{1}{\cosh(t_2)})(\boldsymbol{e}_2 \otimes \boldsymbol{e}_2 + \boldsymbol{e}_3 \otimes \boldsymbol{e}_3) \tag{243}$$

$$+ \sqrt{3}\tanh(t_2)(\boldsymbol{e}_1 \otimes \boldsymbol{e}_2 + \boldsymbol{e}_2 \otimes \boldsymbol{e}_1)\Bigg] \tag{244}$$

A plot of the three nonzero stress components is given in the following graph ($\sigma_f = 200\text{MPa}$, solid line: T_{12}, dashed line: T_{22}, dotted line: T_{11}).



Process 2: Again, we note that the process parameters are chosen such that the shear test brings the material to $\phi = 0$ at $t = 1$, for which $\boldsymbol{E} = \frac{\sigma_f}{2\sqrt{3}G}(\boldsymbol{e}_1 \otimes \boldsymbol{e}_2 + \boldsymbol{e}_2 \otimes \boldsymbol{e}_1)$. Note that $\boldsymbol{E}$ is deviatoric. Thus in an elastic

process we have $\boldsymbol{T} = \boldsymbol{T}' = 2G\frac{\sigma_f}{2\sqrt{3}G}(\boldsymbol{e}_1 \otimes \boldsymbol{e}_2 + \boldsymbol{e}_2 \otimes \boldsymbol{e}_1) = \frac{\sigma_f}{\sqrt{3}}(\boldsymbol{e}_1 \otimes \boldsymbol{e}_2 + \boldsymbol{e}_2 \otimes \boldsymbol{e}_1)$. In the following step we impose $\dot{\boldsymbol{E}} = \frac{1}{\mathrm{s}}\frac{\sigma_f}{E}\boldsymbol{e}_1 \otimes \boldsymbol{e}_1$, which needs to be decomposed into the deviatoric and dilatoric parts,

$$\dot{\boldsymbol{E}}' = \frac{1}{\mathrm{s}}\frac{2\sigma_f}{3E}\boldsymbol{e}_1 \otimes \boldsymbol{e}_1 - \frac{1}{\mathrm{s}}\frac{\sigma_f}{3E}(\boldsymbol{e}_2 \otimes \boldsymbol{e}_2 + \boldsymbol{e}_3 \otimes \boldsymbol{e}_3), \tag{245}$$

$$\dot{\boldsymbol{E}}^\circ = \frac{1}{\mathrm{s}}\frac{\sigma_f}{3E}\boldsymbol{I}. \tag{246}$$

The same argumentation as in the first process shows that the second step deforms the material plastically. Thus, the stresses are integrated by

$$\dot{\boldsymbol{T}}' = 2G\Big[\frac{1}{\mathrm{s}}\frac{2\sigma_f}{3E}\boldsymbol{e}_1 \otimes \boldsymbol{e}_1 - \frac{1}{\mathrm{s}}\frac{\sigma_f}{3E}(\boldsymbol{e}_2 \otimes \boldsymbol{e}_2 + \boldsymbol{e}_3 \otimes \boldsymbol{e}_3) \tag{247}$$

$$- \frac{3}{2\sigma_f^2}\Big(\boldsymbol{T}' \cdot\cdot (\frac{1}{\mathrm{s}}\frac{2\sigma_f}{3E}\boldsymbol{e}_1 \otimes \boldsymbol{e}_1 - \frac{1}{\mathrm{s}}\frac{\sigma_f}{3E}(\boldsymbol{e}_2 \otimes \boldsymbol{e}_2 + \boldsymbol{e}_3 \otimes \boldsymbol{e}_3))\Big)\boldsymbol{T}'\Big] \tag{248}$$

$$\tag{249}$$

With $G = \frac{E}{2(1+\nu)}$ we have

$$\dot{\boldsymbol{T}}' = \frac{1}{1+\nu}\frac{1}{\mathrm{s}}\Big[\frac{2\sigma_f}{3}\boldsymbol{e}_1 \otimes \boldsymbol{e}_1 - \frac{\sigma_f}{3}(\boldsymbol{e}_2 \otimes \boldsymbol{e}_2 + \boldsymbol{e}_3 \otimes \boldsymbol{e}_3) - \frac{1}{2\sigma_f}\Big(2T'_{11} - (T'_{22} + T'_{33})\Big)\boldsymbol{T}'\Big], \tag{250}$$

which gives the following ODE system:

$$\dot{T}'_{11} = \frac{1}{1+\nu}\frac{1}{\mathrm{s}}\left[\frac{2\sigma_f}{3} - \frac{1}{2\sigma_f}\Big(2T'_{11} - (T'_{22} + T'_{33})\Big)T'_{11}\right] \tag{251}$$

$$\dot{T}'_{ii} = \frac{1}{1+\nu}\frac{1}{\mathrm{s}}\left[-\frac{\sigma_f}{3} - \frac{1}{2\sigma_f}\Big(2T'_{11} - (T'_{22} + T'_{33})\Big)T'_{ii}\right] \qquad \text{for } (i,j) = \{(2,2),(3,3)\} \tag{252}$$

$$\dot{T}'_{ij} = \frac{1}{1+\nu}\frac{1}{\mathrm{s}}\left[-\frac{1}{2\sigma_f}\Big(2T'_{11} - (T'_{22} + T'_{33})\Big)T'_{ij}\right] \qquad \text{for } (i,j) = \{(1,2),(1,3),(2,3)\} \tag{253}$$

We see that T_{22} and T_{33} do not differ, neither by initial conditions (both equal to zero) nor by their ODE. Thus, taking $T_{22} = T_{33}$, we consider the subsystem

$$\dot{T}'_{11} = \frac{1}{1+\nu}\frac{1}{\mathrm{s}}\left[\frac{2\sigma_f}{3} - \frac{1}{\sigma_f}(T'^2_{11} - T'_{22}T'_{11})\right] \tag{254}$$

$$\dot{T}'_{22} = \frac{1}{1+\nu}\frac{1}{\mathrm{s}}\left[-\frac{\sigma_f}{3} - \frac{1}{\sigma_f}(T'_{22}T'_{11} - T'^2_{22})\right] \tag{255}$$

The latter system is solved by

$$T'_{11} = \frac{2\sigma_f}{3}\tanh\left(\frac{\mathrm{t}_2}{1+\nu}\right) \tag{256}$$

$$T'_{22} = \frac{-\sigma_f}{3}\tanh\left(\frac{\mathrm{t}_2}{1+\nu}\right) = T'_{33}, \tag{257}$$

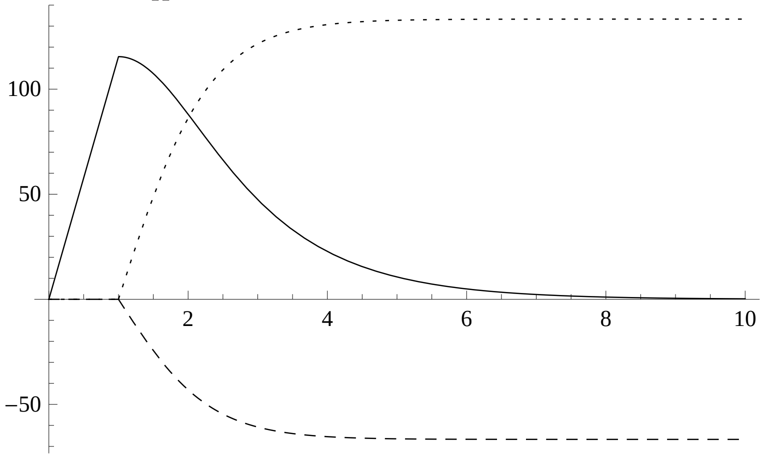
where again the time $t_2 = t - 1\mathrm{s}$ is used. With this result we can go for T'_{ij} with $(i,j) = \{(1,2),(1,3),(2,3)\}$, which is

$$T'_{ij} = \frac{c_{ij}}{\cosh(\frac{\mathrm{t}_2}{1+\nu})}. \tag{258}$$

Again, T'_{23} and T'_{13} remain zero due to their initial values, while $c_{12} = \frac{\sigma_f}{\sqrt{3}}$ results for the initial value of T'_{12}. For the last step, the volumetric part of $\boldsymbol{T}$ is not constant:

$$\boldsymbol{T}^\circ = 3K\dot{\boldsymbol{E}}^\circ t_2. \tag{259}$$

$\boldsymbol{T}^{\circ}$ grows monotonically. For a better separation between the elastic and plastic part of the process, the following graph gives only the variation of the stress deviator over t ($\sigma_f = 200$, $\nu = 0.3$, solid line: T'_{12}, dashed line: T'_{22}, dotted line: T'_{11}):



Homework 1: With the definitions of the hyperbolic sinus, cosine and tangens,

$$\sinh t = \frac{e^t - e^{-t}}{2} \tag{260}$$

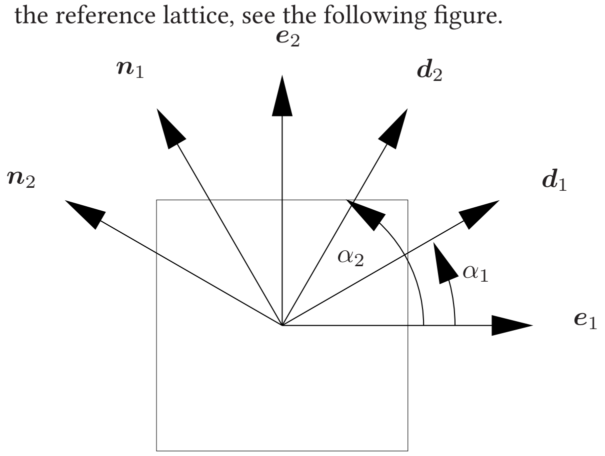
$$\cosh t = \frac{e^t + e^{-t}}{2} \tag{261}$$

$$\tanh t = \frac{\sinh t}{\cosh t} \tag{262}$$

review whether the solutions given here are in fact solutions of the corresponding ODE.
Homework 2: Give an explanation for the saturation behaviour as $t_2 \to \infty$.
Remark: The idea for this task came from Sven Kassbohm, private communication.

9 Crystal plasticity

Consider a rigid-plastic single crystal with two slip systems, which is subjected to an isochoric deformation which is plane in the $\boldsymbol{e}_3$-direction, realized by a constant velocity gradient. The initial lattice (at $t = 0$) corresponds to the reference lattice, see the following figure.



Task: Determine the system of equations which governs the shear in both slip systems and the lattice rotation. Give a solution for symmetric $\boldsymbol{L}$. Examine the system for the case that $\alpha_2 = \alpha_1 + \pi/2$. Solve the system for a shear test ($\boldsymbol{L} = \dot{\gamma}_0 \boldsymbol{e}_1 \otimes \boldsymbol{e}_2$) and an isochoric tension test ($\boldsymbol{L} = \dot{\gamma}_0(\boldsymbol{e}_1 \otimes \boldsymbol{e}_1 - \boldsymbol{e}_2 \otimes \boldsymbol{e}_2)$) numerically for different angles α_1 and α_2 and discuss the results.

Solution: First of all, the overall deformation gradient is given by

$$\boldsymbol{F} = \boldsymbol{F}_\mathrm{e} \boldsymbol{P}^{-1}. \tag{263}$$

Secondly, by demanding a rigid-plastic material behaviour, the elastic part $\boldsymbol{F}_\mathrm{e}$ must be

$$\boldsymbol{F}_\mathrm{e} \in \mathit{Orth}^+. \tag{264}$$

Thirdly, the deformation gradient is by

$$\boldsymbol{L} = \dot{\boldsymbol{F}} \boldsymbol{F}^{-1} \tag{265}$$

linked to the velocity gradient. And last but not least, the rate of slip is by

$$-\boldsymbol{P}^{-1} \dot{\boldsymbol{P}} = \sum_{i=1}^{n} \dot{\gamma}_i \boldsymbol{d}_i \otimes \boldsymbol{n}_i \tag{266}$$

linked to the rate of the plastic transformation, where the index i runs over all slip systems (see the script). In this equation, the lattice vectors are constant reference lattice vectors. We can put this four equations together by taking the time derivative of $\boldsymbol{F}$ (product rule), plugging this into $\boldsymbol{L} = \dot{\boldsymbol{F}} \boldsymbol{F}^{-1}$, and set $\boldsymbol{F}_\mathrm{e} = \boldsymbol{Q}$, which gives

$$\boldsymbol{L} = \dot{\boldsymbol{Q}} \boldsymbol{Q}^T + \boldsymbol{Q} (\boldsymbol{P}^{-1})^{\boldsymbol{\cdot}} \boldsymbol{P} \boldsymbol{Q}^T. \tag{267}$$

By considering the time derivative

$$\dot{\boldsymbol{I}} = \boldsymbol{0} = (\boldsymbol{P}^{-1} \boldsymbol{P})^{\boldsymbol{\cdot}} = (\boldsymbol{P}^{-1})^{\boldsymbol{\cdot}} \boldsymbol{P} + \boldsymbol{P}^{-1} \dot{\boldsymbol{P}} \tag{268}$$

we find that

$$(\boldsymbol{P}^{-1})^{\boldsymbol{\cdot}} \boldsymbol{P} = -\boldsymbol{P}^{-1} \dot{\boldsymbol{P}} = \sum_{i=1}^{n} \dot{\gamma}_i \boldsymbol{d}_i \otimes \boldsymbol{n}_i, \tag{269}$$

i.e. eq. (267) becomes

$$\boldsymbol{L} = \dot{\boldsymbol{Q}}\boldsymbol{Q}^T + \boldsymbol{Q}\sum_{i=1}^{n}\dot{\gamma}_i \boldsymbol{d}_i \otimes \boldsymbol{n}_i \boldsymbol{Q}^T. \tag{270}$$

It is clear that the lattice spin must take place inside of the $\boldsymbol{e}_3$-plane. Thus, we can parametrize $\boldsymbol{Q}$ by

$$\boldsymbol{Q} = \begin{bmatrix} c & s & 0 \\ -s & c & 0 \\ 0 & 0 & 1 \end{bmatrix} \boldsymbol{e}_i \otimes \boldsymbol{e}_j, \tag{271}$$

where we use the abbreviations $c = \cos\phi$ and $s = \sin\phi$. With this we get easily

$$\dot{\boldsymbol{Q}}\boldsymbol{Q}^T = \begin{bmatrix} 0 & \dot{\phi} & 0 \\ -\dot{\phi} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \boldsymbol{e}_i \otimes \boldsymbol{e}_j. \tag{272}$$

With the abbreviations $c_1 = \cos\alpha_1$, $s_1 = \sin\alpha_1$, $c_2 = \cos\alpha_2$, $s_2 = \sin\alpha_2$ we can give $\boldsymbol{d}_i$ and $\boldsymbol{n}_i$ as well with respect to the constant basis $\boldsymbol{e}_i$,

$$\boldsymbol{d}_i = c_i \boldsymbol{e}_1 + s_i \boldsymbol{e}_2, \tag{273}$$
$$\boldsymbol{n}_i = -s_i \boldsymbol{e}_1 + c_i \boldsymbol{e}_2. \tag{274}$$

Putting all together, we can give the components of eq. (270) with respect to the basis $\boldsymbol{e}_i \otimes \boldsymbol{e}_j$, where it is sufficient to consider the upper left 2×2-matrix:

$$\begin{bmatrix} L_{11} & L_{12} \\ L_{21} & -L_{11} \end{bmatrix} = \begin{bmatrix} 0 & \dot{\phi} \\ -\dot{\phi} & 0 \end{bmatrix} + \begin{bmatrix} c & s \\ -s & c \end{bmatrix} \left(\dot{\gamma}_1 \begin{bmatrix} -s_1 c_1 & c_1^2 \\ -s_1^2 & s_1 c_1 \end{bmatrix} + \dot{\gamma}_2 \begin{bmatrix} -s_2 c_2 & c_2^2 \\ -s_2^2 & s_2 c_2 \end{bmatrix} \right) \begin{bmatrix} c & -s \\ s & c \end{bmatrix}. \tag{275}$$

Here we have considered that $\boldsymbol{L}$ must be deviatoric and plane in the $\boldsymbol{e}_3$-plane, i.e. there remain only three independent components. It becomes now evident that we can not consider more than two slip systems: Otherwise we would have more than three functions to be determined (now $\phi(t), \gamma_1(t), \gamma_2(t)$), but still only three equations[1]. The latter equation is summarized to

$$\begin{bmatrix} L_{11} & L_{12} \\ L_{21} & -L_{11} \end{bmatrix} = \\ \begin{bmatrix} -\frac{1}{2}(\dot{\gamma}_1 \sin(2(\alpha_1 - \phi)) + \dot{\gamma}_2 \sin(2(\alpha_2 - \phi))) & \dot{\gamma}_1 \cos^2(\alpha_1 - \phi) + \dot{\gamma}_2 \cos^2(\alpha_2 - \phi) + \dot{\phi} \\ -\dot{\gamma}_1 \sin^2(\alpha_1 - \phi) - \dot{\gamma}_2 \sin^2(\alpha_2 - \phi) - \dot{\phi} & \frac{1}{2}(\dot{\gamma}_1 \sin(2(\alpha_1 - \phi)) + \dot{\gamma}_2 \sin(2(\alpha_2 - \phi))) \end{bmatrix} \tag{276}$$

We immediately note that the right handside 22-component is the negative of the right handside 11-component, as it is the case on the left handside, i.e. comparing the coefficients 11 and 22 gives the same equation. We have, indeed, only three independent equations. The right handside is traceless due to the fact that shear deformations (and superpositions of shear deformations) are isochoric.
We need a little bit more time to note what happens in the case $\alpha_2 = \alpha_1 + \pi/2$. With the relations $\cos(x+\pi/2) = -\sin(x)$, $\sin(x+\pi/2) = \cos(x)$, $\sin(x+\pi) = -\sin(x)$ and $\sin^2(x) + \cos^2(x) = 1$ we can reformulate the latter matrix equation for the case $\alpha_2 = \alpha_1 + \pi/2$ to

$$\begin{bmatrix} L_{11} & L_{12} \\ L_{21} & -L_{11} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}\sin(2(\alpha_1 - \phi))(\dot{\gamma}_1 - \dot{\gamma}_2) & \cos^2(\alpha_1 - \phi)(\dot{\gamma}_1 - \dot{\gamma}_2) + \dot{\phi} + \dot{\gamma}_2 \\ -\sin^2(\alpha_1 - \phi)(\dot{\gamma}_1 - \dot{\gamma}_2) - \dot{\phi} - \dot{\gamma}_2 & \frac{1}{2}\sin(2(\alpha_1 - \phi))(\dot{\gamma}_1 - \dot{\gamma}_2). \end{bmatrix} \tag{277}$$

[1]This is known as the Taylor problem. In three dimensions a deviatoric $\boldsymbol{L}$ has 8 independent components. A general $\boldsymbol{Q}$ is parametrized by 3 angles, which leaves 5 linear independent slip systems. In general a crystal has more than 5 slip systems, which leaves the problem as it is considered here with a variety of solutions. This problem of uniqueness is overcome by taking into account elastic strains, which give rise to stresses (material law), which are used to determine a Schmid stress in each slip system. By postulating a slip rate-stress relation (more material law), the plastic deformation can be uniquely related to the stress state, which is uniquely related to the actual state of plastic and elastic deformation.

One now notes that we can substitute $\dot{\gamma}_1 - \dot{\gamma}_2 =: \Delta\dot{\gamma}$ and $\dot{\phi} + \dot{\gamma}_2 =: \dot{\phi}^*$, which leaves a system of three differential equations with effectively two undetermined functions. It is not to expect that we can find two functions that satisfy three independent equations. Thus, we must exclude the case $\alpha_2 = \alpha_1 + \pi/2$ from our considerations.

The analytical solution for symmetric $\boldsymbol{L}$. Considering the skew part of eq. (276) gives

$$W_{12} = \dot{\phi} + {}^1\!/\!{}_2(\dot{\gamma}_1 + \dot{\gamma}_2). \tag{278}$$

By considering the symmetric part we have

$$\boldsymbol{D} = \frac{\dot{\gamma}_1}{2}\boldsymbol{Q}(\boldsymbol{d}_1 \otimes \boldsymbol{n}_1 + \boldsymbol{n}_1 \otimes \boldsymbol{d}_1)\boldsymbol{Q}^T + \frac{\dot{\gamma}_2}{2}\boldsymbol{Q}(\boldsymbol{d}_2 \otimes \boldsymbol{n}_2 + \boldsymbol{d}_2 \otimes \boldsymbol{n}_2)\boldsymbol{Q}^T. \tag{279}$$

$\boldsymbol{D}$ is symmetric and traceless, hence there are two independent components D_{11} and D_{12}. Presuming that $\boldsymbol{d}_1 \otimes \boldsymbol{n}_1 + \boldsymbol{n}_1 \otimes \boldsymbol{d}_1$ and $\boldsymbol{d}_2 \otimes \boldsymbol{n}_2 + \boldsymbol{n}_2 \otimes \boldsymbol{d}_2$ are linearly independent, the latter is a linear system for $\dot{\gamma}_{1,2}$. Its solutions are

$$\dot{\gamma}_1 = -\frac{2}{\sin(2(\alpha_1 - \alpha_2))}(D_{11}\cos(2(\alpha_2 - \phi)) + D_{12}\sin(2(\alpha_2 - \phi))) \tag{280}$$

$$\dot{\gamma}_2 = \frac{2}{\sin(2(\alpha_1 - \alpha_2))}(D_{11}\cos(2(\alpha_1 - \phi)) + D_{12}\sin(2(\alpha_1 - \phi))). \tag{281}$$

We note that in case of $\alpha_2 - \alpha_1 = n\pi/2,\ n \in \mathbb{Z}$, the solution is not defined. We can insert the latter two equations into eq. (278), which gives an ODE only for ϕ.

$$W_{12} = \frac{1}{\sin(\alpha_1 - \alpha_2)}(D_{12}\cos(\alpha_1 + \alpha_2 - 2\phi) - D_{11}\sin(\alpha_1 + \alpha_2 - 2\phi)) + \dot{\phi} \tag{282}$$

This ODE can be simplified considerably by noting that we can use trigonometric functions to parametrize the set of possible deviatoric $\boldsymbol{D}$ in the following manner,

$$\boldsymbol{D} = \frac{\|\boldsymbol{D}\|}{\sqrt{2}}\begin{bmatrix}\cos\delta & \sin\delta \\ \sin\delta & -\cos\delta\end{bmatrix}\boldsymbol{e}_i \otimes \boldsymbol{e}_j, \tag{283}$$

i.e. we change the independent components D_{11} and D_{22} for $\|\boldsymbol{D}\|$ and δ. The ODE reads then, after employing a trigonometric relation

$$W_{12} = -K\sin(\alpha_1 + \alpha_2 - \delta - 2\phi) + \dot{\phi}, \qquad K = \frac{\|\boldsymbol{D}\|}{\sqrt{2}\cos(\alpha_1 - \alpha_2)}. \tag{284}$$

One notes that in case of $\alpha_2 - \alpha_1 = \pi/2 + n\pi,\ n \in \mathbb{Z}$, K is not defined. The latter ODE can be solved for symmetric and constant $\boldsymbol{L}$, i.e. for $W_{12} = 0$ and constant K:

$$\dot{\phi} = K\sin(\alpha_1 + \alpha_2 - \delta - 2\phi) \tag{285}$$

With $\beta = \alpha_1 + \alpha_2 - \delta - 2\phi$ and consequently $\dot{\phi} = -\dot{\beta}/2$ we have

$$\dot{\beta} = -2K\sin(\beta) \tag{286}$$

$$\frac{\mathrm{d}\beta}{\sin\beta} = -2K\mathrm{d}t \tag{287}$$

Integration gives

$$\ln(\tan(\beta/2)) = -2Kt + C \tag{288}$$

$$\phi = -\arctan(\exp(-2\mathrm{Kt} + \mathrm{C})) + \frac{\alpha_1 + \alpha_2 - \delta}{2}. \tag{289}$$

C is determined from the initial condition $\phi(0) = 0$, i.e.

$$C = \ln(\tan(\frac{\alpha_1 + \alpha_2 - \delta}{2})). \tag{290}$$

We can substitute ϕ in

$$\dot{\gamma}_1 = -\sqrt{2}\|\boldsymbol{D}\|\frac{\cos(2\alpha_2 - \delta - 2\phi)}{\sin(2(\alpha_1 - \alpha_2))} \tag{291}$$

$$\dot{\gamma}_2 = \sqrt{2}\|\boldsymbol{D}\|\frac{\cos(2\alpha_1 - \delta - 2\phi)}{\sin(2(\alpha_1 - \alpha_2))}, \tag{292}$$

which gives, after invoking some trigonometric relations,

$$\dot{\gamma}_1 = -\frac{K}{\sin(\alpha_1 - \alpha_2)} \cos\left[\alpha_1 - \alpha_2 - 2\text{arccot}\left[\exp[2Kt]\cot[(\alpha_1 + \alpha_2 - \delta)/2]\right]\right] \tag{293}$$

$$\dot{\gamma}_2 = \frac{K}{\sin(\alpha_1 - \alpha_2)} \cos\left[\alpha_1 - \alpha_2 + 2\text{arccot}\left[\exp[2Kt]\cot[(\alpha_1 + \alpha_2 - \delta)/2]\right]\right]. \tag{294}$$

Luckily, the latter can be integrated, which gives solutions for γ_1 and γ_2:

$$\gamma_1 = a + b + \tilde{\gamma}_1 \tag{295}$$

$$\gamma_2 = a - b + \tilde{\gamma}_2 \tag{296}$$

$$a = -\,\text{arctan}\left[e^{2Kt}\text{cot}\left[\frac{1}{2}(\alpha_1 + \alpha_2 - \delta)\right]\right] \tag{297}$$

$$b = \frac{1}{2}\text{cot}[\alpha_1 - \alpha_2]\left(2Kt - \ln\left[1 + e^{4Kt} + \left(e^{4Kt} - 1\right)\cos[\alpha_1 + \alpha_2 - \delta]\right]\right) \tag{298}$$

From the initial conditions $\gamma_i = 0$ at $t = 0$ we obtain $\tilde{\gamma}_i$,

$$\tilde{\gamma}_1 = -\tilde{a} - \tilde{b} \tag{299}$$

$$\tilde{\gamma}_2 = -\tilde{a} + \tilde{b} \tag{300}$$

$$\tilde{a} = -\text{arctan}\left(\text{cot}((\alpha_1 + \alpha_2 - \delta)/2)\right) \tag{301}$$

$$\tilde{b} = -\text{ln}(2)\text{cot}(\alpha_1 - \alpha_2)/2 \tag{302}$$

We can examine the behaviour as t tends to infinity: $\exp(Kt + C)$ tends either to 0 or to ∞, depending on the sign of K, which means that

- **(1)** ϕ tends to $(\alpha_1 + \alpha_2 - \delta)/2$ or $(\alpha_1 + \alpha_2 - \delta - \pi)/2$.

A little bit of geometric consideration shows that ϕ tends always to the bisection of the smaller one of the angles that $\boldsymbol{d}_1$ and $\pm\boldsymbol{d}_2$ form. Since ϕ tends to a stationary value, it follows from eqs. (291) and (292) that

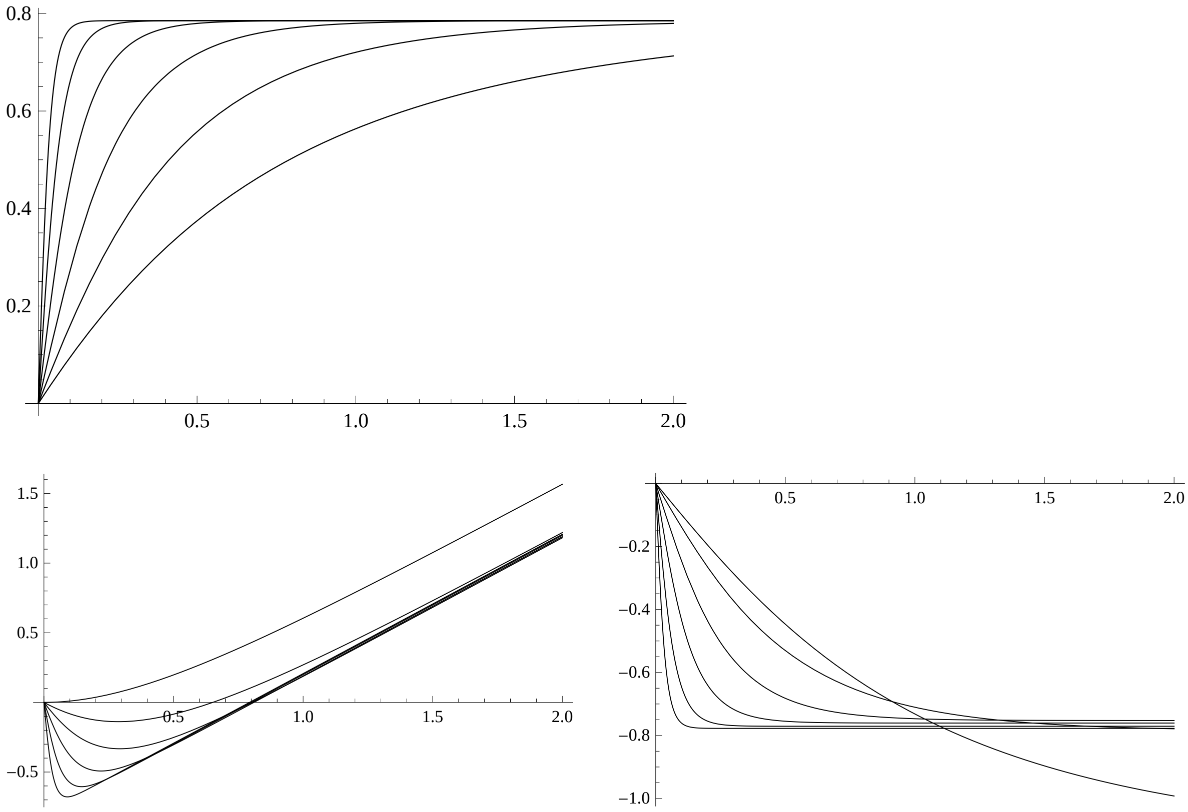
- **(2)** $\dot{\gamma}_i$ tend to constant values of different sign but the same absolute value, thus γ_i approach a linear function, and
- **(3)** the influence of δ on $\dot{\gamma}_i$ tends to zero as t tends to infinity.

The analytical results have been compared to numerical results, which are obtained easily by employing a numerical time integration scheme, and found to practically coincide. Thus, the analytical findings are very likely to be correct. However, we are due to the assumption $\boldsymbol{L} = \boldsymbol{D}$ not able to give results for a simple shear test analytically, and must resort to numerical approximations.

Shear and elongation test. Let us now look at the numerical solutions for different conditions. We take $\alpha_1 = \pi/4$ as constant, and $\alpha_2 = \alpha_1 + {}^{\pi}/_{2}(1 - {}^{1}/_{2^i})$. By this, we can examine the behaviour of the system as $\alpha_2 \to \alpha_1 + \pi/2$ by taking $i \to \infty$. The following Mathematica script solves the system numerically, and plots the solution.

```
Remove["Global`*"]

(* Initialize empty lists for the plots *)

plotsphi = {};
plotsgamma1 = {};
plotsgamma2 = {};

(* Build ODE system *)

Q = {{Cos[phi[t]], Sin[phi[t]]}, {-Sin[phi[t]], Cos[phi[t]]}};
d1 = {Cos[alpha1], Sin[alpha1]};
n1 = {-Sin[alpha1], Cos[alpha1]};
d2 = {Cos[alpha2], Sin[alpha2]};
n2 = {-Sin[alpha2], Cos[alpha2]};
M1 = g1'[t]*Table[d1[[i]]*n1[[j]], {i, 1, 2}, {j, 1, 2}];
M2 = g2'[t]*Table[d2[[i]]*n2[[j]], {i, 1, 2}, {j, 1, 2}];
RechteSeite =
 FullSimplify[D[Q, t].Transpose[Q] + Q.(M1 + M2).Transpose[Q]]

(* Solve ODE system for different angles *)

For[i = 1, i < 7, i++,
  alpha1 = Pi/4;
  alpha2 = alpha1 + Pi/2*(1 - 1/2^i);
  gammadot = 1;
  eq1 = 0 == RechteSeite[[1, 1]];
  eq2 = gammadot == RechteSeite[[1, 2]];
  eq3 = 0 == RechteSeite[[2, 1]];
  tende = 2;
  s = NDSolve[{eq1, eq2, eq3, phi[0] == 0, g1[0] == 0,
     g2[0] == 0}, {phi[t], g1[t], g2[t]}, {t, 0, tende}];
  plotsphi =
   Append[plotsphi,
    Plot[Evaluate[phi[t] /. s], {t, 0, tende}, PlotRange -> All,
     PlotStyle -> {Black}, PlotLabel -> "phi[t]"]];
  plotsgamma1 =
   Append[plotsgamma1,
    Plot[Evaluate[g1[t] /. s], {t, 0, tende}, PlotRange -> All,
     PlotStyle -> {Black}, PlotLabel -> "gamma1[t]"]];
  plotsgamma2 =
   Append[plotsgamma2,
    Plot[Evaluate[g2[t] /. s], {t, 0, tende}, PlotRange -> All,
     PlotStyle -> {Black}, PlotLabel -> "gamma2[t]"]];
  ];

(* Draw plots *)

Show[plotsphi]
Show[plotsgamma1]
Show[plotsgamma2]
```

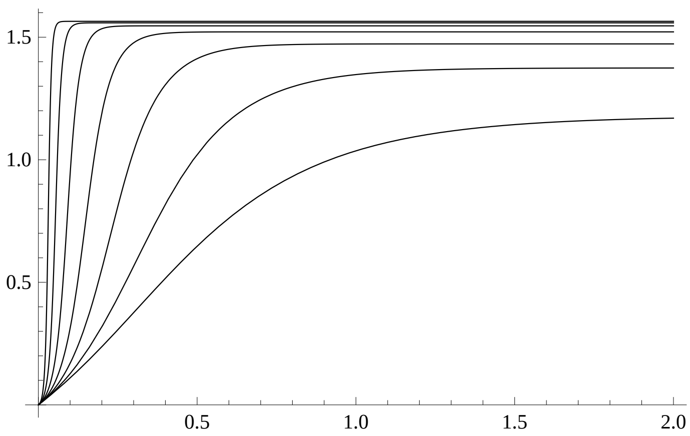
This are the result for the shear test $\boldsymbol{L} = \dot{\gamma}_0 \boldsymbol{e}_1 \otimes \boldsymbol{e}_2$ with $\dot{\gamma}_0 = {}^1\!/\mathrm{s}$ for different α_2 ($i = 1 \dots 6$):

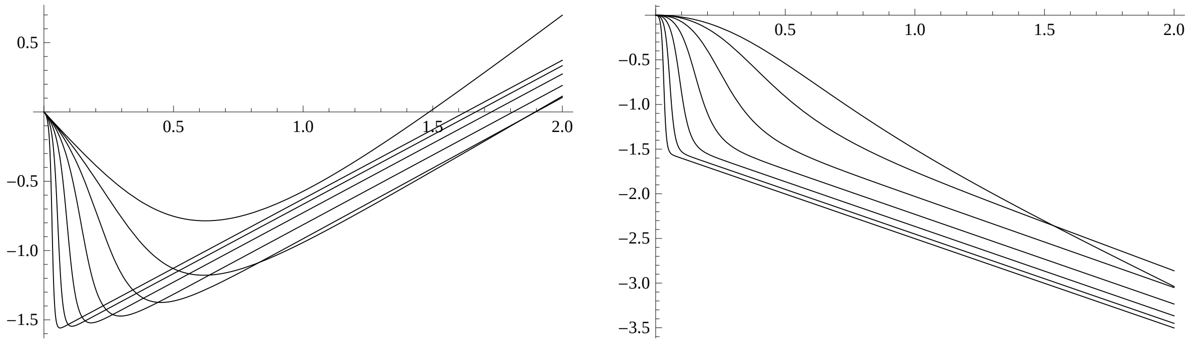


Upper figure: $\phi(t)$, lower left: $\gamma_1(t)$, lower right: $\gamma_2(t)$.

The results indicate that during the shear test, the lattice undergoes a rotation which tends asymptotically to an absolute rotation angle of $\pi/4$. This rotation leaves slip system 1 perfectly aligned to accommodate the global shear deformation alone, i.e. the spatial lattice vectors $\boldsymbol{Q}\boldsymbol{d}_1 \to \boldsymbol{e}_1$ and $\boldsymbol{Q}\boldsymbol{n}_1 \to \boldsymbol{e}_2$. This tendency is more pronounced as $\alpha_2 \to \alpha_1 + \pi/2$.

This are the result for the elongation test $\boldsymbol{L} = \dot{\gamma}_0(\boldsymbol{e}_1 \otimes \boldsymbol{e}_1 - \boldsymbol{e}_2 \otimes \boldsymbol{e}_2)$ with $\dot{\gamma}_0 = {}^{1}\!/\mathrm{s}$ for different α_2 $(i = 1 \ldots 6)$:





Upper figure: $\phi(t)$, lower left: $\gamma_1(t)$, lower right: $\gamma_2(t)$.

The results indicate that during the elongation test, the lattice undergoes a rotation which tends asymptotically to an absolute rotation angle of $^1/_2(\alpha_1 + \alpha_2)$. This rotation leaves the slip systems aligned symmetrically with respect to the elongation direction, i.e. both slip systems contribute equally to the imposed elongation test. This tendency is more pronounced as $\alpha_2 \to \alpha_1 + \pi/2$.

One can imagine that, since this toy problem is hardly manageable analytically, a real crystal (with more slip systems, employing elasticity, Schmid law and flow rule to avoid the Taylor problem), or even a polycrystal (grain interaction, solution of the boundary value problem) requires numerical methods or strong restrictions.

10 Mises-Plasticity at large strains

We take the decomposition of

$$\boldsymbol{F} = \boldsymbol{F}_e \boldsymbol{F}_p \tag{303}$$

as the starting point, which can be derived from **the isomorphy of the elastic law**. The elastic law is given through **Ciarlet's isotropic strain energy**

$$w = w^\circ(\boldsymbol{F}_e) + w'(\boldsymbol{F}_e) = \underbrace{\frac{\lambda}{4}(\mathrm{I\!I\!I} - \ln \mathrm{I\!I\!I} - 1) + \frac{\mu}{2}\ln \mathrm{I\!I\!I}}_{w^\circ} + \underbrace{\frac{\mu}{2}\mathrm{I}}_{w'}\,, \tag{304}$$

which is close to the St.-Venant-Kirchhoff strain energy, but polyconvex. The invariants are that of $\boldsymbol{C}_e$ (resp. $\boldsymbol{B}_e$). w depends not on the rotational part of the polar decomposition of $\boldsymbol{F}_e$. w' contains the change of the elastic energy due to a shape change, w° is the change of the elastic energy due to a volume change. The stress power is

$$l = \frac{1}{\rho}\boldsymbol{T} \cdot\cdot \boldsymbol{L}. \tag{305}$$

With the multiplicative decomposition of $\boldsymbol{F}$, the velocity gradient is

$$\boldsymbol{L} = \dot{\boldsymbol{F}}\boldsymbol{F}^{-1} = \underbrace{\dot{\boldsymbol{F}}_e\boldsymbol{F}_e^{-1}}_{\boldsymbol{L}_e} + \boldsymbol{F}_e \underbrace{\dot{\boldsymbol{F}}_p\boldsymbol{F}_p^{-1}}_{\boldsymbol{L}_p} \boldsymbol{F}_e^{-1}. \tag{306}$$

This allows to identify an elastic and a plastic part of the stress power. The elastic part is just the change of the elastic energy $\dot{w}$, the second summand is the plastic dissipation. From this we can identify immediately the elastic law,

$$\dot{w} = \frac{1}{\rho}\boldsymbol{T} \cdot\cdot \dot{\boldsymbol{F}}_e\boldsymbol{F}_e^{-1} = \frac{\partial w(\boldsymbol{F}_e)}{\partial \boldsymbol{F}_e} \cdot\cdot \dot{\boldsymbol{F}}_e = \frac{\partial w(\boldsymbol{F}_e)}{\partial \boldsymbol{F}_e}\boldsymbol{F}_e^T \cdot\cdot \dot{\boldsymbol{F}}_e\boldsymbol{F}_e^{-1}, \tag{307}$$

where comparing coefficients gives

$$\boldsymbol{T} = \rho\frac{\partial w(\boldsymbol{F}_e)}{\partial \boldsymbol{F}_e}\boldsymbol{F}_e^T. \tag{308}$$

Next, we need a flow criterion. Mises suggested a **limiting of the elastic shape-change energy**,

$$\phi(\boldsymbol{F}_e) = w'(\boldsymbol{F}_e) - w'_{\text{max}}. \tag{309}$$

When $\phi < 0$ the process is elastic, and $\dot{\boldsymbol{F}}_p = \boldsymbol{0}$. If $\phi = 0$ (flow condition) and $\dot{\phi}|_{\boldsymbol{F}_p=\text{const}} > 0$ (loading condition), $\dot{\boldsymbol{F}}_p$ evolves such that $\dot{\phi} = 0$ (consistency condition). This is still not sufficient to specify a flow direction. We assume the **maximum plastic dissipation**. Maximizing the plastic dissipation gives immediately the flow direcion: $\boldsymbol{T} \cdot\cdot \boldsymbol{F}_e\boldsymbol{L}_p\boldsymbol{F}_e^{-1}$ is a scalar product in a vector space, which is maximal when both vectors have the same direction and sense of direction. However, $\boldsymbol{T}$ is symmetric, hence we can only conclude the symmetric part of $\boldsymbol{F}_e\boldsymbol{L}_p\boldsymbol{F}_e^{-1}$. With rate independence, this gives

$$(\boldsymbol{F}_e\dot{\boldsymbol{F}}_p\boldsymbol{F}_p^{-1}\boldsymbol{F}_e^{-1})_{\text{sym}} = \frac{\lambda}{\rho}\boldsymbol{T}, \tag{310}$$

with the consistency parameter λ to be determine from the consistency condition. We cannot make any statement about the skew part of $\boldsymbol{F}_e\dot{\boldsymbol{F}}_p\boldsymbol{F}_p^{-1}\boldsymbol{F}_e^{-1}$. It corresponds to a rotation of the intermediate placement, i.e. an expansion with $\boldsymbol{Q}\boldsymbol{Q}^T$, which is neither plastic nor elastic. In terms of elastic isomorphisms, the elastic reference law and the isomorphism $\boldsymbol{P}$ are changed accordingly. Since the skew part is arbitrary, we can simply drop the symmetrization and insert the elastic law,

$$\dot{\boldsymbol{F}}_p \boldsymbol{F}_p^{-1} = \lambda \boldsymbol{F}_e^{-1} \frac{\partial w(\boldsymbol{F}_e)}{\partial \boldsymbol{F}_e} \boldsymbol{C}_e. \tag{311}$$

We consider only volume preserving plastic deformations, so $\boldsymbol{L}_p$ is deviatoric,

$$\dot{\boldsymbol{F}}_p \boldsymbol{F}_p^{-1} = \lambda (\boldsymbol{F}_e^{-1} \frac{\partial w(\boldsymbol{F}_e)}{\partial \boldsymbol{F}_e} \boldsymbol{C}_e)'. \tag{312}$$

We determine λ by the consistency condition,

$$0 = \dot{\phi} = \frac{\partial w'(\boldsymbol{F}_e)}{\partial \boldsymbol{F}_e} \cdot\cdot \dot{\boldsymbol{F}}_e = \frac{\partial w'(\boldsymbol{F}_e)}{\partial \boldsymbol{F}_e} \boldsymbol{F}_e^T \cdot\cdot \dot{\boldsymbol{F}}_e \boldsymbol{F}_e^{-1}, \tag{313}$$

with

$$\dot{\boldsymbol{F}}_e \boldsymbol{F}_e^{-1} = \boldsymbol{L} - \boldsymbol{F}_e \dot{\boldsymbol{F}}_p \boldsymbol{F}_p^{-1} \boldsymbol{F}_e^{-1} = \boldsymbol{L} - \lambda \boldsymbol{F}_e (\boldsymbol{F}_e^{-1} \frac{\partial w(\boldsymbol{F}_e)}{\partial \boldsymbol{F}_e} \boldsymbol{C}_e)' \boldsymbol{F}_e^{-1}, \tag{314}$$

which gives

$$\lambda = \frac{\frac{\partial w'(\boldsymbol{F}_e)}{\partial \boldsymbol{F}_e} \boldsymbol{F}_e^T \cdot\cdot \boldsymbol{L}}{\frac{\partial w'(\boldsymbol{F}_e)}{\partial \boldsymbol{F}_e} \boldsymbol{F}_e^T \cdot\cdot \boldsymbol{F}_e (\boldsymbol{F}_e^{-1} \frac{\partial w(\boldsymbol{F}_e)}{\partial \boldsymbol{F}_e} \boldsymbol{C}_e)' \boldsymbol{F}_e^{-1}}. \tag{315}$$

However, an analytical expression for λ is in general useless, since one mostly needs a numerical time integration, which ensures the consistency condition at discrete time steps. Then λ depends on the choice of the time integration scheme.