Chapter 8
The Eigenmodes in Isotropic Strain Gradient Elasticity

Rainer Glüge, Jan Kalisch & Albrecht Bertram

Abstract We present the spectral decomposition of the isotropic stiffness hexadic that appears in Mindlin's strain gradient elasticity, where the kinematic variable is the second gradient of the displacement field. It turns out that four distinct eigenmodes appear, two of which are universal for all isotropic strain gradient materials, and two depend on an additional material parameter. With the aid of the harmonic decomposition, general interpretations of the eigenmodes can be given. Further, the material parameters are related to commonly employed special cases, namely the cases tabulated in Neff et al (2009) and isotropic gradient elasticity of Helmholtz type.

Key words: Strain gradient plasticity, Spectral decomposition, Stiffness hexadic

8.1 Introduction

It is well known that classical elasticity can not account for size effects that are observed in very small structures (Liebold and Müller, 2013). Mostly, the specific stiffness of fine structures is increased. It is also well known that one can overcome this shortcoming by including a strain gradient dependence in the elastic energy. The isotropic extension of linear elasticity has been given by Mindlin (1964). It involves a sixth-order stiffness tensor with five independent parameters, which relates

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Rainer Glüge *(corresponding author)* & Albrecht Bertram
Lehrstuhl für Festigkeitslehre, Institut für Mechanik, Fakultät für Maschinenbau, Otto-von-Guericke-Universität Magdeburg, Universitätsplatz 2, 39106 Magdeburg, Germany,
e-mail: gluege@ovgu.de, albrecht.bertram@ovgu.de

Jan Kalisch
Graduiertenkolleg 1554 "Micro-Macro-Interactions in Structured Media and Particle Systems", Otto-von-Guericke-Universität Magdeburg, Universitätsplatz 2, 39106 Magdeburg, Germany,
e-mail: j.kalisch@ovgu.de

the strain gradient to the hyperstress tensor. The aim of the present work is to give a spectral decomposition of this hexadic. The eigenmodes of this hexadic may be interpreted geometrically, similar to the eigenmodes of the well-known Hooke-tetradic of classical isotropic linear elasticity, which are volume- and shape changing deformations. The eigenmodes can be interpreted in terms of displacement fields, curvature, volume- and shape changing deformations, local rotations, and so on. By this, the eigenvalues and hence the 5 independent parameters of the hexadic become interpretable. For convenience, we have included a conversion of some special cases from the literature to the five independent parameters in Mindlin's strain gradient elasticity. A more general account to strain gradient theories on this topic can be found in Bertram (2015).

The present article builds on isotropic strain gradient elasticity (Mindlin and Eshel, 1968) and decomposition and representation theorems for isotropic tensors of arbitrary order, as found in Golubitsky et al (1988); Zheng and Zou (2000); Olive and Auffray (2014); Auffray et al (2013).

8.1.1 Notation

We prefer a direct notation, but make use of Einstein's summation convention (implicit summation over pairs of indices) whenever necessary. Scalars, vectors, second- and higher-order tensors are denoted by italic letters (like a or A), bold minuscules (like $\mathbf{a}$), bold majuscules (like $\mathbf{A}$), and blackboard bold majuscules (like $\mathbb{A}$), respectively. Moreover, $\{\mathbf{e}_i\}$ denotes an orthonormal basis. The single contraction and the dyadic product are denoted by $\cdot$ and $\otimes$, respectively. Multiple contractions act in the same sense on either tensor, e. g., $(\mathbf{a}\otimes\mathbf{b}\otimes\mathbf{c})\cdot\cdot(\mathbf{d}\otimes\mathbf{e}) = (\mathbf{b}\cdot\mathbf{d})(\mathbf{c}\cdot\mathbf{e})\,\mathbf{a}$.

For groups and vector spaces, we use calligraphic letters, such as $\mathscr{H}$ for the space of right subsymmetric third-order tensors $\mathbb{H}$, for which $\mathbb{H}\cdot\cdot\mathbf{A} = \mathbb{H}\cdot\cdot\mathbf{A}^T$ holds. In particular, we denote harmonic tensor spaces of order i by $\mathscr{H}_i$. $\mathbf{I}$, $\mathbb{I}_S$, $\mathbb{I}$ and $\boldsymbol{\varepsilon}$ denote the identities on vectors, symmetric second-order tensors, subsymmetric third-order tensors and the third-order permutation tensor.

8.2 Isotropic Stiffness Hexadic

In general, the elastic energy of a strain gradient material is written in terms of the symmetric second-order strain tensor $\mathbf{E} = \mathrm{sym}(\mathbf{u}\otimes\nabla) = \mathrm{sym}(\mathbf{H})$ and a third-order tensor with one subsymmetry as the strain gradient variable $\mathbb{H}$. The latter may be the gradient of the strain $\mathbf{E}\otimes\nabla$, or the second gradient of the displacement $\mathbf{u}(\mathbf{x}_0,t)$. Mindlin (1964) refers to these two choices as strain gradient elasticity of form 1 and form 2. In any case, the third-order tensor $\mathbb{H}$ has one subsymmetry (left or right), and therefore only 18 independent components. It is interesting to note that these symmetries have different origins. In one case, the subsymmetry is a purely mathematical consequence (Schwartz' theorem), in the other case it comes from purging the rotations from the first gradient deformation measure.

Here, we will use the second gradient of the displacement $\mathbf{u}$ as the strain gradient variable, i.e., we take

$$\mathbb{H} = \mathbf{u} \otimes \nabla \otimes \nabla. \tag{8.1}$$

Approaching the elastic energy density as a quadratic form, we get

$$w = \frac{1}{2} C_{ijkl} E_{ij} E_{kl} + C_{ijklm} E_{ij} H_{klm} + \frac{1}{2} C_{ijklmn} H_{ijk} H_{lmn} \tag{8.2}$$

w.r.t. an ONB. Here appear the fourth-, fifth- and sixth-order stiffness tensors $\overset{\langle 4\rangle}{\mathbb{C}}$, $\overset{\langle 5\rangle}{\mathbb{C}}$ and $\overset{\langle 6\rangle}{\mathbb{C}}$, all of which are determined only up to some subsymmetric part that is due to the symmetries of the involved variables $\mathbf{E}$ and $\mathbb{H}$. Further, $\overset{\langle 4\rangle}{\mathbb{C}}$ and $\overset{\langle 6\rangle}{\mathbb{C}}$ have the principle symmetry, since they are multiplied twice with the same variable,

$$C_{ijkl} = C_{klij} = C_{jikl} = C_{ijlk}, \tag{8.3}$$
$$C_{ijklm} = C_{jiklm} = C_{ijkml}, \tag{8.4}$$
$$C_{ijklmn} = C_{lmnijk} = C_{ikjlmn} = C_{ijklnm}. \tag{8.5}$$

Presuming these index symmetries alone, $\overset{\langle 4\rangle}{\mathbb{C}}$, $\overset{\langle 5\rangle}{\mathbb{C}}$ and $\overset{\langle 6\rangle}{\mathbb{C}}$ have 21, 108, and 171 independent components, respectively. However, these numbers can be drastically reduced when material symmetries are taken into account. A particular case is isotropy. The components of any isotropic tensor can be given in terms of Kronecker- and Levi-Civita symbols w.r.t. an ONB. Due to the index symmetries of $\overset{\langle 4\rangle}{\mathbb{C}}$, $\overset{\langle 5\rangle}{\mathbb{C}}$ and $\overset{\langle 6\rangle}{\mathbb{C}}$ and the anti-symmetry of the Levi-Civita symbol, only Kronecker-deltas appear, which means that $\overset{\langle 5\rangle}{\mathbb{C}} = \mathbb{O}$ in case of centrosymmetric isotropy. For $\overset{\langle 4\rangle}{\mathbb{C}}$ and $\overset{\langle 6\rangle}{\mathbb{C}}$, we have the well-known representations (see, e.g., Mindlin (1964, 1965); Lazar and Maugin (2005); dell'Isola et al (2009); Bertram and Forest (2014)). In Mindlin's notation with $\eta_{ijk} := u_{k,ij}$, the strain gradient energy density is

$$w = 2c_2 \eta_{kii} \eta_{kjj} + c_4 \eta_{ijk} \eta_{ijk} + 2c_3 \eta_{ijk} \eta_{jki} + \frac{c_5}{2} \eta_{jji} \eta_{kki} + 2c_1 \eta_{iik} \eta_{kjj}. \tag{8.6}$$

This can be written as a quadratic form $u_{i,jk} C_{ijklmn} u_{l,mn}/2$ with a stiffness hexadic

$$\overset{\langle 6\rangle}{\mathbb{C}} = \big[c_1(\delta_{jk}\delta_{im}\delta_{nl} + \delta_{jk}\delta_{in}\delta_{ml} + \delta_{ji}\delta_{kl}\delta_{mn} + \delta_{jl}\delta_{ik}\delta_{mn}) \tag{8.7}$$
$$+\ c_2(\delta_{ji}\delta_{km}\delta_{nl} + \delta_{jm}\delta_{ki}\delta_{nl} + \delta_{ji}\delta_{kn}\delta_{ml} + \delta_{jn}\delta_{ik}\delta_{ml}) \tag{8.8}$$
$$+\ c_3(\delta_{jm}\delta_{kl}\delta_{in} + \delta_{jl}\delta_{in}\delta_{km} + \delta_{jn}\delta_{im}\delta_{kl} + \delta_{jl}\delta_{im}\delta_{nk}) \tag{8.9}$$
$$+\ c_4(\delta_{jn}\delta_{il}\delta_{km} + \delta_{jm}\delta_{kn}\delta_{il}) \tag{8.10}$$

$$+\ c_5\delta_{jk}\delta_{il}\delta_{mn}\big]\mathbf{e}_i\otimes\mathbf{e}_j\otimes\mathbf{e}_k\otimes\mathbf{e}_l\otimes\mathbf{e}_m\otimes\mathbf{e}_n \tag{8.11}$$

We summarize the different combinations of Kronecker symbols that belong to each parameter c_i with the basis $\{\mathbf{e}_i\otimes\mathbf{e}_j\otimes\mathbf{e}_k\otimes\mathbf{e}_l\otimes\mathbf{e}_m\otimes\mathbf{e}_n\}$ to five base hexadics $\{\mathbb{B}_i\}$, such that

$$\overset{\langle 6\rangle}{\mathbb{C}} = \sum_{i=1}^{5} c_i\mathbb{B}_i. \tag{8.12}$$

The metric of the basis $\{\mathbb{B}_i\}$ is

$$\mathbb{B}_i\cdots\cdots\mathbb{B}_j = \begin{bmatrix} 168 & 96 & 96 & 24 & 36\\ 96 & 192 & 72 & 48 & 12\\ 96 & 72 & 192 & 48 & 12\\ 24 & 48 & 48 & 72 & 18\\ 36 & 12 & 12 & 18 & 27 \end{bmatrix}. \tag{8.13}$$

We observe that $\frac{1}{2}\mathbb{B}_4$ maps every subsymmetric triadic onto itself, that $\frac{1}{3}\mathbb{B}_5$ maps every tensor of the form $\mathbf{v}\otimes\mathbf{I}$ onto itself, and that $\frac{1}{8}\mathbb{B}_2$ maps every tensor of the form $\mathbf{I}\otimes\mathbf{v}$ into its right subsymmetric part.

8.2.1 An Orthogonal Basis

Before turning to the spectral decomposition, a more suitable basis is introduced

$$\tilde{\mathbb{B}}_1 := -\frac{1}{15}(\mathbb{B}_1+\mathbb{B}_2+\mathbb{B}_5)+\frac{1}{6}(\mathbb{B}_3+\mathbb{B}_4), \tag{8.14}$$

$$\tilde{\mathbb{B}}_2 := \frac{1}{12}(2\mathbb{B}_1-\mathbb{B}_2-2\mathbb{B}_3+4\mathbb{B}_4-4\mathbb{B}_5), \tag{8.15}$$

$$\tilde{\mathbb{B}}_3 := \frac{1}{60}(6\mathbb{B}_1-9\mathbb{B}_2+16\mathbb{B}_5), \tag{8.16}$$

$$\tilde{\mathbb{B}}_4 := \frac{1}{6\sqrt{5}}(3\mathbb{B}_1-4\mathbb{B}_5), \tag{8.17}$$

$$\tilde{\mathbb{B}}_5 := \frac{1}{20}(-2\mathbb{B}_1+3\mathbb{B}_2+8\mathbb{B}_5). \tag{8.18}$$

The metric of this basis is diagonal with $\tilde{\mathbb{B}}_i\cdots\cdots\tilde{\mathbb{B}}_i=(7,5,6,6,6)$. The components of $\overset{\langle 6\rangle}{\mathbb{C}}$ with respect to this basis are

$$\tilde{c}_1 := 2(c_4-c_3), \tag{8.19}$$

$$\tilde{c}_2 := 4c_3+2c_4, \tag{8.20}$$

$$\tilde{c}_3 := \frac{1}{6}(12c_1-16c_2+2c_3+9c_5), \tag{8.21}$$

$$\tilde{c}_4 := \frac{2\sqrt{5}}{3}\,(3c_1 + 2c_2 + 2c_3)\,, \tag{8.22}$$

$$\tilde{c}_5 := \frac{1}{2}(4c_1 + 8c_2 + 2c_3 + 4c_4 + 3c_5). \tag{8.23}$$

8.2.2 Eigenvalues and Projectors.

In terms of the latter basis $\{\tilde{\mathbb{B}}_i\}$ and components $\tilde{c}_i$, the spectral decomposition of $\overset{\langle 6\rangle}{\mathbb{C}}$ is given by

$$\overset{\langle 6\rangle}{\mathbb{C}} = \sum_{i=1}^{4} \lambda_i \mathbb{P}_i \tag{8.24}$$

with the eigenvalues

$$\lambda_1 = \tilde{c}_1, \tag{8.25}$$

$$\lambda_2 = \tilde{c}_2, \tag{8.26}$$

$$\lambda_3 = \tilde{c}_5 + c_r, \tag{8.27}$$

$$\lambda_4 = \tilde{c}_5 - c_r \tag{8.28}$$

with

$$c_r = \sqrt{\tilde{c}_3^2 + \tilde{c}_4^2} \tag{8.29}$$

and the eigenprojectors

$$\mathbb{P}_1 = \tilde{\mathbb{B}}_1, \tag{8.30}$$

$$\mathbb{P}_2 = \tilde{\mathbb{B}}_2, \tag{8.31}$$

$$\mathbb{P}_3(\kappa) = \frac{1}{2}(\tilde{\mathbb{B}}_5 + \frac{\tilde{c}_3}{c_r}\tilde{\mathbb{B}}_3 + \frac{\tilde{c}_4}{\tilde{c}_r}\tilde{\mathbb{B}}_4), \tag{8.32}$$

$$\mathbb{P}_4(\kappa) = \frac{1}{2}(\tilde{\mathbb{B}}_5 - \frac{\tilde{c}_3}{c_r}\tilde{\mathbb{B}}_3 - \frac{\tilde{c}_4}{\tilde{c}_r}\tilde{\mathbb{B}}_4) \tag{8.33}$$

with

$$\cos\kappa = \frac{\tilde{c}_3}{c_r} \qquad \Leftrightarrow \qquad \sin\kappa = \frac{\tilde{c}_4}{c_r}. \tag{8.34}$$

For the spectral decomposition, the representation of $\overset{\langle 6\rangle}{\mathbb{C}}$ with the dimensionless parameter κ and the four eigenvalues is more convenient than with the parameters $\{c_1, c_2, c_3, c_4, c_5\}$ or $\{\tilde{c}_1, \tilde{c}_2, \tilde{c}_3, \tilde{c}_4, c_r\}$. One can check that

$$\mathbb{P}_3(\kappa) = \mathbb{P}_4(\kappa + \pi), \tag{8.35}$$

$$\lambda_3(\kappa) = \lambda_4(\kappa + \pi) \tag{8.36}$$

holds, i.e., it is reasonable to restrict κ to the interval $[0, \pi)$. The metric of the projectors is diagonal with $\mathbb{P}_i \cdot \cdots \cdot \mathbb{P}_i = (7, 5, 3, 3)$, thus the multiplicities of the eigenvalues are 7, 5, 3 and 3. Further, we have the projector properties

$$\mathbb{P}_i \cdots \mathbb{P}_j = \begin{cases} \mathbb{P}_i \text{ if } i = j \\ \mathbb{O} \text{ if } i \neq j \end{cases} \tag{8.37}$$

$$\sum_{i=1}^{4} \mathbb{P}_i = \mathbb{I}, \tag{8.38}$$

where $\mathbb{I}$ is the sixth-order identity tensor on triads with the right subsymmetry. These equations resemble those of the spectral decomposition of a transversely isotropic stiffness tetradic (see Appendix A of Kalisch and Glüge (2015)), which also has in general five independent components and four distinct eigenvalues.

The above formulae are convenient when one knows the parameters $c_{1,2,3,4,5}$, and seeks the eigenvalues and the third and fourth eigenprojector. The other way around, the coefficients $c_{1,2,3,4,5}$ are given by

$$c_1 = (10\lambda_1 - 4\lambda_2 - 3(\lambda_3 + \lambda_4) + 3(\lambda_3 - \lambda_4)(\cos(\kappa) + \sqrt{5}\sin(\kappa)))/60 \tag{8.39}$$

$$c_2 = (-10\lambda_1 - 8\lambda_2 + 9(\lambda_3 + \lambda_4) + 9(-\lambda_3 + \lambda_4)\cos(\kappa))/120 \tag{8.40}$$

$$c_3 = (-\lambda_1 + \lambda_2)/6 \tag{8.41}$$

$$c_4 = (2\lambda_1 + \lambda_2)/6 \tag{8.42}$$

$$c_5 = (-5\lambda_1 - \lambda_2 + 3(\lambda_3 + \lambda_4) + (\lambda_3 - \lambda_4)(2\cos(\kappa) - \sqrt{5}\sin(\kappa)))/15 \tag{8.43}$$

in terms of $\{\lambda_{1,2,3,4}, \kappa\}$.

8.3 The Eigenmodes and the Harmonic Decomposition

The latter result becomes clearer from the point of view of the harmonic decomposition of a third-order tensor with one subsymmetry (Golubitsky et al, 1988; Zheng and Zou, 2000; Olive and Auffray, 2014). The third and fourth projector – more precisely: the parameter κ – distinguish a specific decomposition of the first-order harmonic contribution, which is discussed next.

The space of all second gradients $\mathbb{H} = \mathbf{u} \otimes \nabla \otimes \nabla$ is subsequently denoted by $\mathscr{H}$. By virtue of the harmonic decomposition a tensor is decomposed into a sum of mutually orthogonal tensors,

$$\mathbb{H} = \sum_{i=1}^{N} \mathbb{H}_i, \tag{8.44}$$

$$0 = \mathbb{H}_i \cdots \mathbb{H}_j, \qquad i \neq j. \tag{8.45}$$

These correspond to the eigentensors of $\overset{\langle 6 \rangle}{\mathbb{C}}$, where N is the number of different eigenvalues. Each $\mathbb{H}_i$ is related to a *harmonic tensor* $\overset{\langle n \rangle}{\mathbb{H}_i}$ by virtue of an isotropic linear mapping $\mathbb{L}_i$

$$\mathbb{H}_i = \overset{\langle 3+n \rangle}{\mathbb{L}}_i \underbrace{\cdots}_{n \text{ dots}} \overset{\langle n \rangle}{\mathbb{H}_i}. \tag{8.46}$$

The order n of the harmonic tensors does not exceed that of the decomposed tensor. The harmonic tensor spaces are denoted by $\mathscr{H}_i$. Their dimensions are

$$dimension \mathscr{H}_i = 2i + 1, \tag{8.47}$$

which is due to the fact that all elements from $\mathscr{H}_i$ are completely symmetric, and all possible index contractions (like H_{ijj}) are zero.

On the whole, the tensor space $\mathscr{H}$ is decomposed into the direct sum ($\oplus$) of mutually orthogonal subspaces. These subspaces are closed under the action of the Rayleigh product with an orthogonal tensor $\mathbf{Q}$, which can be considered as a rotation of $\mathbb{H}$ by $\mathbf{Q}$. The Rayleigh product is defined as

$$\mathbf{Q} * (H_{ijk} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k) = H_{ijk} (\mathbf{Q} \cdot \mathbf{e}_i) \otimes (\mathbf{Q} \cdot \mathbf{e}_j)(\mathbf{Q} \cdot \mathbf{e}_k), \tag{8.48}$$

whereas the closedness under its action is

$$\mathbf{Q} * \mathbb{H} \in \mathscr{H}_i \Leftrightarrow \mathbb{H} \in \mathscr{H}_i \tag{8.49}$$

for all proper orthogonal tensors $\mathbf{Q}$. A further decomposition without loss of this property is not possible, which is why this decomposition is sometimes referred to as irreducible.

The harmonic decomposition can be thought of as the diagonalization of a matrix. The matrix originates from the action of the group of all proper orthogonal tensors on the tensor space (rotation of tensors by means of the Rayleigh product). Subspaces for harmonic spaces of equal order form block matrices on the main diagonal, the dimension of which corresponds to the number of subspaces of equal order. If we define additional orthogonal decompositions, we can diagonalize these block matrices as well. It is shown below that the angle κ parametrizes such an additional decomposition in the present case.

For sufficiently smooth fields $\mathbf{u}$, the respective tensor $\mathbb{H}$ can be represented by a linear combination of products of the form

$$\mathbb{H} = \sum_{i=1\ldots3; j=1\ldots6} C_{ij} \, \mathbf{e}_i \otimes \mathbf{E}_j, \tag{8.50}$$

where $\{\mathbf{e}_i\}$ and $\{\mathbf{E}_j\}$ are orthonormal bases in the three-dimensional Euclidean space and the space of symmetric second-order tensors, respectively. The harmonic decomposition of these spaces is given by $\mathscr{H}_1$ and $\mathscr{H}_0 \oplus \mathscr{H}_2$, respectively. The three-dimensional space cannot be decomposed into harmonic subspaces, hence it is represented by the three-dimensional space $\mathscr{H}_1$. The six-dimensional space of symmetric second-order tensors is decomposed into the well known spherical and deviatoric symmetric parts, the first is one-dimensional and corresponds to $\mathscr{H}_0$, and the second is five-dimensional and corresponds to $\mathscr{H}_2$.

Similar to the decomposition (8.50), the space $\mathscr{H}$ can be constructed as the dyadic product of the form

$$\mathscr{H} = \mathscr{H}_1 \otimes (\mathscr{H}_0 \oplus \mathscr{H}_2). \tag{8.51}$$

With the Clebsch-Gordan rule (Golubitsky et al, 1988)

$$\mathscr{H}_m \otimes \mathscr{H}_n = \bigoplus_{k=|m-n|}^{m+n} \mathscr{H}_k \tag{8.52}$$

we obtain

$$\begin{aligned} \mathscr{H} &\cong \mathscr{H}_1 \otimes (\mathscr{H}_0 \oplus \mathscr{H}_2) && (8.53)\\ &= (\mathscr{H}_1 \otimes \mathscr{H}_0) \oplus (\mathscr{H}_1 \otimes \mathscr{H}_2) && (8.54)\\ &= \mathscr{H}_1 \oplus \mathscr{H}_1 \oplus \mathscr{H}_2 \oplus \mathscr{H}_3 && (8.55)\\ &= \mathscr{H}_3 \oplus \mathscr{H}_2 \oplus \mathscr{H}_1^{\oplus 2}. && (8.56) \end{aligned}$$

Thus, we get two three-dimensional, one five-dimensional and one seven-dimensional subspace, altogether forming the 18-dimensional space of third-order tensors with one symmetry.

The harmonic decomposition is unique regarding the number and the dimensionality of the subspaces. However, when two equally-dimensioned subspaces appear, there is an arbitrariness in the isomorphisms that connect $\mathbb{H}_i$ and $\overset{\langle n\rangle}{\mathbb{H}}_i$. In our representation, this arbitrariness corresponds to the angle κ that determines the direction of the two eigenprojectors $\mathbb{P}_3$ and $\mathbb{P}_4$ of the eigenvalues λ_3 and λ_4, each having the multiplicity 3. The specifications of Eq. (8.46) are

$$\begin{aligned} \mathbb{H}_1 &= \mathbb{H}_1, && (8.57)\\ \mathbb{H}_2 &= \boldsymbol{\varepsilon} \cdot \mathbf{H}_2, && (8.58)\\ \mathbb{H}_3 &= \mathbf{h}_3 \cdot (\cos(\kappa/2)\mathbb{P}_{4/1} + \sin(\kappa/2)\mathbb{P}_{4/2}/\sqrt{5}), && (8.59)\\ \mathbb{H}_4 &= \mathbf{h}_4 \cdot (-\sin(\kappa/2)\mathbb{P}_{4/1} + \cos(\kappa/2)\mathbb{P}_{4/2}/\sqrt{5}). && (8.60) \end{aligned}$$

On both sides of these equations, the index indicates the ordering of the eigenspaces. The $\mathbb{H}_i$ on the left side represent second displacement gradients that are eigentensors in the indexed eigenspaces. The $\mathbb{H}_1$, $\mathbf{H}_2$, $\mathbf{h}_3$ and $\mathbf{h}_4$ (denoted more general as $\overset{\langle n\rangle}{\mathbb{H}}_i$) on the right side are harmonic (fully symmetric and traceless) tensors of order 3,2, 1 and 1, hence having 7, 5, 3 and 3 independent components. The number of these independent components corresponds to the dimension of the eigenspaces. Further, $\mathbb{P}_{4/1,2}$ are the isotropic projectors from the spectral decomposition of isotropic stiffness tetradics with the compression modulus K and the shear modulus G,

$$\overset{\langle 4\rangle}{\mathbb{C}} = 3K\underbrace{\frac{1}{3}\mathbf{I}\otimes\mathbf{I}}_{\mathbb{P}_{4/1}} + 2G\underbrace{\left(\mathbb{I}_S - \frac{1}{3}\mathbf{I}\otimes\mathbf{I}\right)}_{\mathbb{P}_{4/2}}. \tag{8.61}$$

$\mathbb{I}_S = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\mathbf{e}_i\otimes\mathbf{e}_j\otimes\mathbf{e}_k\otimes\mathbf{e}_l$ is the identity on symmetric second-order tensors. With this symbolic representation of the eigenmodes, we can examine their properties by virtue of the traceless and symmetric properties of their corresponding harmonic tensors.

8.3.1 The 7-Dimensional Eigenspace $\mathscr{H}_3$

With $\mathbb{H}_1$ being harmonic, we find the traces and index symmetries

$$u_{i,jj} = 0, \tag{8.62}$$
$$u_{i,ik} = 0, \tag{8.63}$$
$$u_{i,jk} = u_{j,ik}. \tag{8.64}$$

Thus, $\mathbf{u}$ is a harmonic function, and the volumetric strain must be homogeneous. After Helmholtz' representation theorem (Helmholtz, 1858), there exist a scalar field ϕ and a divergence free (solenoidal) vector field $\mathbf{a}$ (Coulomb's gauge) such that

$$\mathbf{u} = \nabla\phi + \nabla\times\mathbf{a}, \qquad \mathbf{a}\cdot\nabla = 0. \tag{8.65}$$

Using $u_{i,i} = \mathbf{u}\cdot\nabla = \Delta\phi$, we find with Eq. (8.64)

$$\nabla(\Delta\,\phi) = \mathbf{o}. \tag{8.66}$$

Equation (8.64) can be rewritten as

$$(\mathbf{u}\times\nabla)\otimes\nabla = \mathbf{O}, \tag{8.67}$$

i.e., the rotational part of $\mathbf{u}$ is homogeneous. Then, the Helmholtz representation and Coulomb's gauge imply

$$(\Delta\,\mathbf{a})\otimes\nabla = \mathbf{O}. \tag{8.68}$$

Given sufficiently smooth fields, Laplacian and gradient commute. Thus, $\Delta\phi = u_{i,i}$ and $\Delta\mathbf{a}$ are homogeneous (Eqs 8.66, 8.68) and $\nabla\phi$ and $\mathbf{a}\otimes\nabla$ are harmonic functions.

In conclusion, the displacement fields that generate eigenstrain-gradients in $\mathscr{H}_3$

- are free from volumetric strain gradients,
- have zero mean curvature of the displacement compontens, and
- the gradient of the axial vector $\mathbf{u}\times\nabla$ vanishes everywhere, i.e., the rotational part of the displacement field is homogeneous.

8.3.2 The 5-Dimensional Eigenspace $\mathscr{H}_2$

For convenience, we drop the index at $\mathbf{H}_2$ in this paragraph. In index notation w.r.t. an ONB we get

$$u_{i,jk} = \frac{1}{2}(\varepsilon_{ijl}H_{lk} + \varepsilon_{ikl}H_{lj}), \tag{8.69}$$

where u_i is a displacement field that produces only strain gradients in the 5-dimensional eigenspace that is isomorphic to $\mathscr{H}_2$.

We cannot directly transfer the traceless and symmetric properties of $\mathbf{H}$ to the displacement gradient, since a summation index is involved in $\mathbf{H}$ but not in $u_{i,jk}$. Taking the two independent traces of $u_{i,jk}$ gives

$$u_{i,jj} = \frac{1}{2}(\varepsilon_{ijl}H_{lj} + \varepsilon_{ijl}H_{lj}) = \varepsilon_{ijl}H_{lj} = 0 \quad \Leftrightarrow \quad \mathrm{axi}(\mathrm{skw}(\mathbf{H})) = \mathbf{o}, \tag{8.70}$$

$$u_{j,jk} = \frac{1}{2}(\varepsilon_{jjl}H_{lk} + \varepsilon_{jkl}H_{lj}) = \frac{1}{2}\varepsilon_{ijl}H_{lj} = 0 \quad \Leftrightarrow \quad \frac{1}{2}\mathrm{axi}(\mathrm{skw}(\mathbf{H})) = \mathbf{o}, \tag{8.71}$$

i.e., they give the same information. The skew part of $\mathbf{H}$ (and hence the axial vector $\mathbf{w}$ =axi(skw($\mathbf{H}$)) implicitly defined by $\mathbf{w}\times\mathbf{x}$ =skw($\mathbf{H}$)$\cdot\mathbf{x}$) is zero by definition. Thus, we find that the eigenstrain gradients of the 5-dimensional eigenspace belong to harmonic displacement fields without volumetric strain gradient, as in the case before. Now we consider

$$(\varepsilon_{nij}\,u_{i,j})_{,k} = \varepsilon_{nij}\,u_{i,jk} \tag{8.72}$$

$$= \varepsilon_{nij}\,(u_{i,jk} - u_{j,ik})/2 \tag{8.73}$$

$$= \varepsilon_{nij}\,(2\varepsilon_{ijm}H_{mk} + \varepsilon_{ikm}H_{mj} - \varepsilon_{jkm}H_{mi})/4 \tag{8.74}$$

$$= (2\varepsilon_{ijn}\varepsilon_{ijm}H_{mk} + \varepsilon_{ijn}\varepsilon_{ikm}H_{mj} - \varepsilon_{jni}\varepsilon_{jkm}H_{mi})/4 \tag{8.75}$$

$$= \delta_{nm}H_{mk} + [(\delta_{jk}\,\delta_{nm} - \delta_{jm}\,\delta_{nk})\,H_{mj} \tag{8.76}$$

$$-(\delta_{nk}\,\delta_{im} - \delta_{nm}\,\delta_{ik}]\,H_{mi})/4 \tag{8.77}$$

$$= H_{nk} + (H_{nk} - \delta_{nk}\,H_{mm} - \delta_{nk}\,H_{mm} + H_{nk})/4 \tag{8.78}$$

$$= 3\,H_{nk}/2. \tag{8.79}$$

In symbolic notation we thus have

$$\mathbf{H} \propto (\mathbf{u} \times \nabla) \otimes \nabla \tag{8.80}$$
$$= (-\Delta\, \mathbf{a}) \otimes \nabla \tag{8.81}$$
$$= -\Delta\, (\mathbf{a} \otimes \nabla). \tag{8.82}$$

H is symmetric and deviatoric. The latter property is in accordance with Coulomb's condition on **a**. The symmetry of **H** implies another constraint on **a**.

$$\Delta\, (\mathbf{a} \otimes \nabla) = \Delta\, (\nabla \otimes \mathbf{a}), \tag{8.83}$$
$$\Leftrightarrow \quad \mathbf{O} = \Delta\, (\mathbf{a} \otimes \nabla - \nabla \otimes \mathbf{a}) \tag{8.84}$$
$$\mathbf{O} = \Delta\, \varepsilon \cdot (\mathbf{a} \times \nabla) \tag{8.85}$$
$$= \varepsilon \cdot (\Delta\, (\mathbf{a} \times \nabla)), \tag{8.86}$$
$$\Leftrightarrow \quad \mathbf{o} = \Delta\, (\mathbf{a} \times \nabla) \tag{8.87}$$
$$= (\Delta\, \mathbf{a}) \times \nabla \tag{8.88}$$

The divergence of Eq. (8.84) provides – by means of Coulomb's gauge

$$\mathbf{o} = (\Delta\, (\mathbf{a} \otimes \nabla - \nabla \otimes \mathbf{a})) \cdot \nabla \tag{8.89}$$
$$= \Delta\, [(\mathbf{a} \otimes \nabla) \cdot \nabla - (\nabla \otimes \mathbf{a}) \cdot \nabla] \tag{8.90}$$
$$= \Delta\, (\Delta\, \mathbf{a} - \nabla\, (\mathbf{a} \cdot \nabla)) \tag{8.91}$$
$$= \Delta\, \Delta\, \mathbf{a}. \tag{8.92}$$

Thus, **a** must be a biharmonic function. In conclusion, the displacement fields that generate eigenstrain-gradients in $\mathscr{H}_2$

- are free from volumetric strain gradients,
- have zero mean curvature of the displacement compontens, and
- the divergence of the gradient of the axial vector $\mathbf{u} \times \nabla$ vanishes everywhere.

These restrictions are weaker (third bullet point) than in case of eigenstrain-gradients of $\mathbb{H}_1$. This is not surprising, as we have less constraints to exploit, namely only one zero trace and one index symmetry, due to $\mathbf{H}_2$ being a second order tensor.

8.3.3 The 3-Dimensional Eigenspaces

Unfortunately $\mathbf{h}_{3,4}$ have no symmetry or zero trace which could be exploited. The third and fourth eigenmode depend on the angle κ, which depends on the coefficients $c_{1,2,3,4,5}$ through Eq. (8.34). Thus, we can determine canonical angles κ by taking one of the c_i as infinite, or consider more general directional limits with fixed ratios between the c_i. In doing so, two special cases emerge, namely when c_2 or c_5 are taken to infinity. In both cases, the third eigenvalue λ_3 becomes infinite, and its eigenprojector $\mathbb{P}_3$ becomes $\frac{1}{8}\mathbb{B}_2$ or $\frac{1}{3}\mathbb{B}_5$, respectively. The angles κ that belong to these materials can be inferred from Eq. (8.34), and one finds

$$c_2 \to \infty: \qquad \cos\kappa \to -\frac{2}{3}, \qquad \mathbb{P}_3 = \frac{1}{8}\mathbb{B}_2, \qquad \lambda_3 \to \infty, \tag{8.93}$$

$$c_5 \to \infty: \qquad \cos\kappa \to 1, \qquad \mathbb{P}_3 = \frac{1}{3}\mathbb{B}_5, \qquad \lambda_3 \to \infty. \tag{8.94}$$

However, we can also adjust κ and the eigenvalues $\lambda_{1,2,3,4}$ independently.

8.3.3.1 The Case $\cos\kappa = -2/3$

The eigentensors of the third and fourth eigenvalue are related to the harmonic tensors $\mathbf{h}_3$ and $\mathbf{h}_4$ through

$$\mathbb{H}_3 = \mathbf{h}_3 \cdot \mathbb{I}_S = \mathrm{sym}_{23}\mathbf{I} \otimes \mathbf{h}_3, \tag{8.95}$$

$$\mathbb{H}_4 = \mathbf{h}_4 \cdot (\mathbb{I}_S - 6\mathbb{P}_{4/1})/\sqrt{5} \tag{8.96}$$

This case is closest to the usual strain decomposition into dilatoric and deviatoric parts. The eigenmodes to the third eigenvalue are gradients of the volumetric strain. The fourth eigenmode does not correspond to a gradient of a deviatoric strain. By considering

$$\cos\kappa = \frac{\tilde{c}_3}{c_r} = -\frac{2}{3}, \tag{8.97}$$

$$\sin\kappa = \frac{\tilde{c}_4}{c_r} = \frac{\sqrt{5}}{3}, \tag{8.98}$$

(remember that $\kappa \in [0, \pi)$), eliminating c_r and summarizing, one finds that this case corresponds to

$$4c_1 + 2c_3 + c_5 = 0. \tag{8.99}$$

8.3.3.2 The Case $\cos\kappa = 1$

The eigentensors of the third and fourth eigenvalue are related to the harmonic tensors $\mathbf{h}_3$ and $\mathbf{h}_4$ through

$$\mathbb{H}_3 = \mathbf{h}_3 \cdot \mathbb{P}_{4/1}, \tag{8.100}$$

$$\mathbb{H}_4 = \mathbf{h}_4 \cdot \mathbb{P}_{4/2}. \tag{8.101}$$

A calculation similar to the symbolic examination of the 5- and 7-dimensional eigenspaces shows that both eigenstrain gradients $\mathbb{H}_3$ and $\mathbb{H}_4$ result from displacement fields with a biharmonic field ϕ in their Helmholtz representations. In terms of c_i, this case corresponds to

$$3c_1 + 2c_2 + 2c_3 = 0. \tag{8.102}$$

8.4 Relation to Other Forms of Strain Gradient Elasticity

For convenience, we summarize the conversion of parameters between the two forms of strain gradient elasticity and for special cases of the first form of strain gradient elasticity (Mindlin and Eshel, 1968). We follow the list given in Neff et al (2009) (Eqs 2.10) and Lazar's proposal of gradient elasticity of Helmholtz type (Po et al, 2014).

8.4.1 Mindlin's Second Form of Strain Gradient Elastictiy

The two forms of strain gradient elasticity (Mindlin and Eshel, 1968) are

$$w_1 = \frac{1}{2}\mathbf{u}\otimes\nabla\otimes\nabla\cdots\mathbb{C}\cdots\mathbf{u}\otimes\nabla\otimes\nabla, \tag{8.103}$$

$$w_2 = \frac{1}{2}\nabla\otimes\mathrm{sym}(\mathbf{u}\otimes\nabla)\cdots\hat{\mathbb{C}}\cdots\nabla\otimes\mathrm{sym}(\mathbf{u}\otimes\nabla), \tag{8.104}$$

where we use the very same base tensors $\mathbb{B}_{1,2,3,4,5}$, but with the parameters $\hat{c}_{1,2,3,4,5}$. The conversion between the two variants is

$$c_1 = \hat{c}_1/2 + \hat{c}_2/2, \tag{8.105}$$

$$c_2 = \hat{c}_1/2 + \hat{c}_2/4 + \hat{c}_5/4, \tag{8.106}$$

$$c_3 = 3\hat{c}_3/4 + \hat{c}_4/4, \tag{8.107}$$

$$c_4 = \hat{c}_3/2 + \hat{c}_4/2, \tag{8.108}$$

$$c_5 = \hat{c}_2. \tag{8.109}$$

Our conversion differs from the one given in Mindlin and Eshel (1968) (Eq. 2.6), since here we considered the components of the stiffness hexadic w.r.t. the base tensors $\mathbb{B}_i$, whereas Mindlin considered the coefficients in the strain gradient energy. The differences are due to symmetrizations, see see Eq. (8.6). Apart from that, the ordering is different.

8.4.2 Common Strain Gradient Extensions

We translate directly the forms in table 2.10 from Neff et al (2009):

Table 8.1 Special cases of strain gradient elasticity translated into the parameter set $c_{1,2,3,4,5}$, where the left column contains the strain energy density, the center column the corresponding parameters c_i and the right column the eigenvalues λ_i and the angle κ.

el. energy w	$c_{1,2,3,4,5}$	$\lambda_{1,2,3,4}, \kappa$
$\|\mathbf{u}\otimes\nabla\otimes\nabla\|^2$	$0,0,0,1,0$	$2,2,2,2,$ arbitrary
$\|\Delta\mathbf{u}\|^2$	$0,0,0,0,2$	$0,0,6,0,0$
$\|\mathrm{sym}(\mathbf{u}\otimes\nabla)\otimes\nabla\|^2$	$0,0,1/4,1/2,0$	$2,1/2,2,1/2,\arccos(1/9)$
$\|\mathrm{devsym}(\mathbf{u}\otimes\nabla)\otimes\nabla\|^2$	$0,-1/6,1/4,1/2,0$	$2,1/2,7/6,0,\arccos(19/21)$
$\|\mathrm{skw}((\mathbf{u}\times\nabla)\otimes\nabla)\|^2$	$-1/2,1/4,0,0,1$	$0,0,3,0,\arccos(-1/9)$
$\|(\mathbf{u}\times\nabla)\times\nabla\|^2$	$-1,1/2,0,0,2$	$0,0,6,0,\arccos(-1/9)$
$\|(\mathbf{u}\cdot\nabla)\nabla\|^2$	$0,1/2,0,0,0$	$0,0,4,0,\arccos(-2/3)$
$\|(\mathbf{u}\times\nabla)\otimes\nabla\|^2$	$0,0,-1/2,1,0$	$0,3,3,0,\arccos(-1/9)$
$\|\mathrm{dev}((\mathbf{u}\times\nabla)\otimes\nabla)\|^2$	$0,0,-1/2,1,0$	$0,3,3,0,\arccos(-1/9)$
$\|\mathrm{sym}((\mathbf{u}\times\nabla)\otimes\nabla)\|^2$	$1/2,-1/4,-1/2,1,-1$	$0,3,0,0,$arbitrary
$\|\mathrm{devsym}((\mathbf{u}\times\nabla)\otimes\nabla)\|^2$	$1/2,-1/4,-1/2,1,-1$	$0,3,0,0,$ arbitrary
$\|\mathrm{sym}(\mathrm{sym}(\mathbf{u}\otimes\nabla)\times\nabla)\|^2$	$1/8,-1/16,-1/8,1/4,-1/4$	$0,3/4,0,0,$ arbitrary

8.4.3 Gradient Elasticity of Helmholtz Type

In order to reduce the number of elasticity constants, Lazar et al (2006) recommend to use

$$C_{ijklmn} = l^2 C_{jkmn}\delta_{il}, \tag{8.110}$$

in the second form (Eq. 8.104), with the fourth-order stiffness tetradic and the additional material parameter l. In case of anisotropic elasticity, the second-order tensor that extends the stiffness tetradic is invariant under the action of the material symmetry group. In case of isotropy and cubic elasticity, this is a multiple of the identity tensor, with the parameter l^2. The conversion to $c_{1,2,3,4,5}$ is

$$c_1 = 0, \tag{8.111}$$
$$c_2 = l^2(K/4 - G/6), \tag{8.112}$$
$$c_3 = l^2 G/4, \tag{8.113}$$
$$c_4 = l^2 G/2, \tag{8.114}$$
$$c_5 = 0. \tag{8.115}$$

Thus, the third and fourth eigenmode depend via κ on the internal length parameter l and the compression and shear moduli K and G. In terms of Mindlin's second form of strain gradient elasticity (see Sect. 8.4.1), we have only two nonzero parameters, namely

$$\hat{c}_1 = 0, \tag{8.116}$$
$$\hat{c}_2 = 0, \tag{8.117}$$
$$\hat{c}_3 = 0, \tag{8.118}$$

$$\hat{c}_4 = Gl^2 = \mu l^2, \tag{8.119}$$

$$\hat{c}_5 = Kl^2 - 2Gl^2/3 = \lambda l^2, \tag{8.120}$$

where the inheritance from the classical isotropic stiffness tetradic with Lamé's constants λ and μ is more obvious.

Acknowledgements Valuable hints from Patrizio Neff and Markus Lazar are gratefully acknowledged.

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